

ORIENTED INTERVAL GREEDOIDS

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ABSTRACT. We propose a definition of an *oriented interval greedoid* that simultaneously generalizes the notion of an oriented matroid and the construction on antimatroids introduced by L. J. Billera, S. K. Hsiao, and J. S. Provan in *Enumeration in convex geometries and associated polytopal subdivisions of spheres* [Discrete Comput. Geom. **39** (2008), no. 1-3, 123–137]. As for oriented matroids, associated to each oriented interval greedoid is a spherical simplicial complex whose face enumeration depends only on the underlying interval greedoid.

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1. INTRODUCTION

Consider a hyperplane arrangement in \mathbb{R}^n , with all of the hyperplanes containing the origin. Intersecting this arrangement with a sphere centred at the origin, one obtains a regular cell decomposition of the sphere. Taking the barycentric subdivision, one obtains a spherical simplicial complex.

Oriented matroids are a generalization of real hyperplane arrangements; the Sphericity Theorem of Folkman and Lawrence [FL78] for oriented matroids says that any oriented matroid induces a certain regular cell decomposition of the sphere (and thus also a spherical simplicial complex) just as hyperplane arrangements do. (Terms not defined in the introduction will be defined later in the paper.)

Billera, Hsiao, and Provan showed in [BHP08] that there is also a certain spherical simplicial complex associated to a *convex geometry* or *antimatroid*. These simplicial complexes are not a special case of the spheres arising from oriented matroids, but they are similar in some respects (see §6.3 in particular).

The goal of this paper is to provide a general theory which includes both of these as special cases. Following a suggestion in [BHP08] (attributed to Anders Björner), our approach is via the notion of interval greedoid. The precise definition appears in the next section, but for now, it suffices to know that interval greedoids are a common generalization of matroids and antimatroids.

In this paper, we define the notion of an oriented interval greedoid. This is an additional structure on top of the interval greedoid structure. For a given interval greedoid, there may be no such additional structure possible, or one, or more than one.

For an interval greedoid which is a matroid, an oriented interval greedoid structure amounts to (the collection of covectors defining) an oriented matroid. In contrast, if the underlying interval greedoid is an antimatroid, it always admits exactly one oriented interval greedoid structure.

Our main result is an analogue of the Sphericity Theorem for oriented interval greedoids, providing a CW-sphere and (by barycentric subdivision) a spherical simplicial complex associated to any oriented interval greedoid. Our proof is based on the proof of the Sphericity Theorem given in [BLVS⁺93]. The spherical simplicial complex associated to the unique oriented structure for an antimatroid, coincides with that constructed by [BHP08].

Along the way, we give versions for oriented interval greedoids of a number of constructions for oriented matroids, such as restriction and contraction.

2. INTERVAL GREEDOIDS

Much of the material in this section, except for §2.6.4, is drawn from [BZ92b] or [KLS91]. The material in §2.6.4 and, by extension, the treatment in §2.6.5, seems to be new.

2.1. Definition. Let E denote a finite set and \mathcal{F} a set of subsets of E . An *interval greedoid* is a pair (E, \mathcal{F}) satisfying the following properties for all $X, Y, Z \in \mathcal{F}$:

- (IG1) If $X \neq \emptyset$, then there exists an $x \in X$ such that $X - x \in \mathcal{F}$.
- (IG2) If $|X| > |Y|$, then there exists an $x \in X \setminus Y$ such that $Y \cup x \in \mathcal{F}$.
- (IG3) If $X \subseteq Y \subseteq Z$ and $e \in E \setminus Z$ with $X \cup e \in \mathcal{F}$ and $Z \cup e \in \mathcal{F}$, then $Y \cup e \in \mathcal{F}$.

The set E is called the *ground set* of the interval greedoid (E, \mathcal{F}) . Elements of \mathcal{F} are called the *feasible sets* of (E, \mathcal{F}) . If \mathcal{F} is a nonempty collection of subsets of E satisfying (IG1), then \mathcal{F} is said to be an *accessible set system*. A *greedoid* is a pair (E, \mathcal{F}) that satisfies (IG1) and (IG2). In the literature, (IG3) is often called the *interval property*.

In the next few sections we present several examples of interval greedoids.

2.2. Example: Matroids (Lower Interval Greedoids). A *matroid* is a pair (E, \mathcal{I}) where E is a finite set and \mathcal{I} is a collection of subsets of E satisfying the following two properties:

- (M1) If $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$.
- (M2) For all $X, Y \in \mathcal{I}$, if $|X| > |Y|$, then there exists an $x \in X \setminus Y$ such that $Y \cup x \in \mathcal{I}$.

Since (M2) is (IG2) and (M1) is a strengthening of (IG1) that implies the interval property (IG3), a matroid (E, \mathcal{I}) is an interval greedoid. Conversely, any greedoid (E, \mathcal{F}) satisfying the following strengthening of (IG3) is a matroid.

- (LIP) Suppose $X, Y \in \mathcal{F}$ with $X \subseteq Y$. If $e \in E \setminus Y$ and $Y \cup e \in \mathcal{F}$, then $X \cup e \in \mathcal{F}$.

The above is called the *interval property without lower bounds*, so a matroid is a *lower interval greedoid*.

Example 2.2.1 (Vector matroids). Let $V = \mathbb{R}^2$, $\vec{x} = (-3, 1)$, $\vec{y} = (2, 1)$ and $\vec{z} = (4, 1)$. See Figure 1. Let \mathcal{I} be the collection of subsets of $E = \{\vec{x}, \vec{y}, \vec{z}\}$ that consist of linearly independent vectors. That is,

$$\mathcal{I} = \left\{ \emptyset, \{\vec{x}\}, \{\vec{y}\}, \{\vec{z}\}, \{\vec{x}, \vec{y}\}, \{\vec{x}, \vec{z}\}, \{\vec{y}, \vec{z}\} \right\}.$$

Then (E, \mathcal{I}) is a matroid. ○

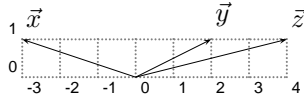


FIGURE 1.

2.3. Example: Antimatroids (Upper Interval Greedoids). Another class of interval greedoids arise from convex geometries.

2.3.1. *Convex geometries.* Just as matroids can be viewed as an abstraction of linear independence of vectors in \mathbb{R}^n , convex geometries can be viewed as an abstraction of convexity of vectors in \mathbb{R}^n . In the following, think of E as a finite subset of \mathbb{R}^n and τ as the convex hull operator: $\tau(A) = \text{conv}(A) \cap E$ for $A \subseteq E$.

A **convex geometry** is a pair (E, τ) , where E is a finite set and $\tau : 2^E \rightarrow 2^E$ is an increasing, monotone and idempotent function, satisfying the following **anti-exchange** axiom.

(AE) If $x, y \notin \tau(X)$, $x \neq y$, and $y \in \tau(X \cup x)$, then $x \notin \tau(X \cup y)$.

The subsets $A \subseteq E$ satisfying $\tau(A) = A$ are called **closed sets** of the convex geometry. The **extreme points** $\text{ext}(A)$ of $A \subseteq E$ are the points $x \in A$ satisfying $x \notin \tau(A \setminus x)$. The extreme points form a *minimal generating set* for the closed sets: if $X \subseteq E$ is a closed set, then $X = \tau(\text{ext}(X))$, and $\text{ext}(X) \subseteq Y$ for all $Y \subseteq E$ satisfying $\tau(Y) = X$ ([BZ92b, Proposition 8.7.2] or [KLS91, Theorem III.1.1]).

Example 2.3.1 (Convex geometries from convexity). The canonical example of a convex geometry is a finite subset $E \subseteq \mathbb{R}^n$ with $\tau(A) = \text{conv}(A) \cap E$, where $\text{conv}(A)$ is the convex hull of the points in A . Then the extreme points of A are precisely the extreme points of the convex hull of A . \circ

2.3.2. *Antimatroids, or upper interval greedoids.* If (E, τ) is a convex geometry, then the complements of the closed sets of E

$$\mathcal{F} = \{E \setminus \tau(A) : A \subseteq E\}$$

are the feasible sets of an interval greedoid on the ground set E . Moreover, (E, \mathcal{F}) satisfies the following **interval property without upper bounds**.

(UIP) Suppose $X, Y \in \mathcal{F}$ with $X \subseteq Y$. If $e \in E \setminus Y$ and $X \cup e \in \mathcal{F}$, then $Y \cup e \in \mathcal{F}$.

If (E, \mathcal{F}) is a greedoid satisfying (UIP), then it is said to be an **upper interval greedoid**, or an **antimatroid**. All upper interval greedoids arise from convex geometries: if (E, \mathcal{F}) is an upper interval greedoid, then the complements of the feasible sets are the closed sets of the convex geometry (E, τ) , where τ is defined for $X \subseteq E$ by

$$\tau(X) = \bigcap_{\substack{X \subseteq Y \subseteq E \\ E \setminus Y \in \mathcal{F}}} Y.$$

In other words, $\tau(X)$ is the smallest set in $\mathcal{F}^c = \{E \setminus Y : Y \in \mathcal{F}\}$ containing X . For a proof of this result, see [KLS91, Theorem III.1.3] or [BZ92b, Proposition 8.7.3].

Example 2.3.2 (Antimatroid from three colinear points). Let x, y, z be three colinear points in the plane, y between x and z , and consider the convex geometry with closure operator $\tau(X) = \text{conv}(X) \cap \{x, y, z\}$ (see Example 2.3.1). The closed sets are the subsets

$$\emptyset, \{x\}, \{y\}, \{z\}, \{x, y\}, \{y, z\}, \{x, y, z\}.$$

Then (E, \mathcal{F}) is an upper interval greedoid, where $E = \{x, y, z\}$ and \mathcal{F} is

$$\mathcal{F} = \left\{ \{x, y, z\}, \{y, z\}, \{x, z\}, \{x, y\}, \{z\}, \{x\}, \emptyset \right\}. \quad \circ$$

Remark 2.3.3. Upper interval greedoids have been studied under several different names including antimatroid, APS-structures, discs, and shelling structures. Some care is required in reading the literature, as some authors have used the term antimatroid for a convex geometry. By antimatroid, we will always mean an upper interval greedoid.

2.4. Example: Interval greedoids from semimodular lattices. Let L be a finite lattice. L is said to be (*lower*) **semimodular** if it has the following property for all $x, y \in L$: if $x \leq z$ and $y \leq z$ for some $z \in L$, then $x \wedge y \leq x$ and $x \wedge y \leq y$. An element $e \in L$ such that $e \neq \hat{1}$ is called **meet-irreducible** if $e = x \wedge y$ implies $x = e$ or $y = e$.

Proposition 2.4.1. *Suppose L is a finite lower semimodular lattice. Let E be the set of meet-irreducible elements of L , and let*

$$\mathcal{F} = \{ \{e_1, e_2, \dots, e_k\} \subseteq E : \hat{1} \geq e_1 \geq (e_1 \wedge e_2) \geq \dots \geq (e_1 \wedge e_2 \wedge \dots \wedge e_k) \}.$$

Then (E, \mathcal{F}) is an interval greedoid.

For a proof of this result see [BZ92b, Theorem 8.8.7].

2.5. Feasible Orderings. Let (E, \mathcal{F}) denote an interval greedoid. Let $X \in \mathcal{F}$. An ordering $x_1 < x_2 < \dots < x_r$ of the elements of $X = \{x_1, x_2, \dots, x_r\}$ is denoted by $X = \{x_1 < x_2 < \dots < x_r\}$. An ordering $X = \{x_1 < x_2 < \dots < x_r\}$ is a **feasible ordering** of X if $\{x_1, \dots, x_i\} \in \mathcal{F}$ for all $1 \leq i \leq r = |X|$. Repeated application of (IG1) shows that every $X \in \mathcal{F}$ has a feasible ordering.

Proposition 2.5.1. *Let (E, \mathcal{F}) be an interval greedoid. Let $X, Y \in \mathcal{F}$ and $|X| > |Y|$. Suppose $X = \{x_1 < \dots < x_r\}$ is a feasible ordering. Then there is a subset $\{x_{i_1} < \dots < x_{i_k}\}$ of $X \setminus Y$ of size $|X| - |Y|$ such that $Y \cup \{x_{i_1}, \dots, x_{i_j}\} \in \mathcal{F}$ for all $1 \leq j \leq k$.*

Proof. Let $x_1 < \dots < x_r$ be a feasible ordering of X and suppose $Y \in \mathcal{F}$ with $|Y| < |X|$. We proceed by induction on $|Y|$. If $|Y| = 0$, then the feasible ordering $x_1 < \dots < x_r$ of X provides the required subset.

Suppose the result holds for all feasible sets of cardinality less than $|Y|$. By (IG1), since $Y \in \mathcal{F}$, there is a $y \in Y$ such that $Y \setminus y \in \mathcal{F}$. Since $|Y \setminus y| < |Y| < |X|$, the induction hypothesis gives the existence of a subset $\{x_{i_1} < \dots < x_{i_k}\}$ of X of size $k = |X| - (|Y| - 1)$ such that $Y \setminus y \cup \{x_{i_1}, \dots, x_{i_j}\} \in \mathcal{F}$ for all $1 \leq j \leq k$.

Since $|Y| < |X| = |(Y \setminus y) \cup \{x_{i_1}, \dots, x_{i_k}\}|$, it follows from repeated application of (IG2) that there exist elements z_j in $\{x_{i_1}, \dots, x_{i_k}\}$ such that $Y \cup \{z_1\}, Y \cup \{z_1, z_2\}, \dots, Y \cup \{z_1, \dots, z_{k-1}\}$ are in \mathcal{F} . Suppose that for each $1 \leq l < k$ the element z_l is chosen to be the first element (with respect to the feasible ordering on X) satisfying $(Y \cup \{z_1, \dots, z_{l-1}\}) \cup z_l \in \mathcal{F}$.

Since $|Y \cup \{z_1, \dots, z_{l-1}\}| < |(Y \setminus y) \cup \{x_{i_1}, \dots, x_{i_{l+1}}\}|$, it follows from (IG2) that there is an element $z \in \{x_{i_1}, \dots, x_{i_{l+1}}\} \setminus \{z_1, \dots, z_{l-1}\}$ such that $(Y \cup \{z_1, \dots, z_{l-1}\}) \cup z \in \mathcal{F}$. The minimality of z_l implies z_l is amongst these elements. That is, $z_l \in \{x_{i_1}, \dots, x_{i_{l+1}}\} \setminus \{z_1, \dots, z_{l-1}\}$.

Let a be the first index for which $z_a \neq x_{i_a}$. Then from the last sentence in the previous paragraph,

$$\begin{aligned} z_a &\in \{x_{i_1}, \dots, x_{i_{a+1}}\} \setminus \{z_1, \dots, z_{a-1}\} \\ &= \{x_{i_1}, \dots, x_{i_{a+1}}\} \setminus \{x_{i_1}, \dots, x_{i_{a-1}}\} \\ &= \{x_{i_a}, x_{i_{a+1}}\}. \end{aligned}$$

Thus, $z_a = x_{i_{a+1}}$.

Suppose $z_b = x_{i_a}$ for some index b . Induction on l gives $z_l = x_{i_{l+1}}$ for all l such that $a < l < b$ since $z_l \in \{x_{i_1}, \dots, x_{i_{l+1}}\} \setminus \{z_1, \dots, z_{l-1}\} = \{x_{i_a}, x_{i_{l+1}}\}$ and $z_l \neq x_{i_a}$. Consider the following three sets: $((Y \setminus y) \cup \{z_1, \dots, z_{a-1}\}) \subseteq (Y \cup \{z_1, \dots, z_{a-1}\}) \subseteq (Y \cup \{z_1, \dots, z_{b-1}\})$. The first is $(Y \setminus y) \cup \{x_{i_1}, \dots, x_{i_{a-1}}\}$, which is in \mathcal{F} by definition of the element x_{i_l} . The latter two sets are in \mathcal{F} by definition of the elements z_l . Substituting $z_l = x_{i_l}$ for $1 \leq l < a$ and $z_l = x_{i_{l+1}}$ for $a \leq l < b$ gives

$$\begin{aligned} ((Y \setminus y) \cup \{x_{i_1}, \dots, x_{i_{a-1}}\}) &\subseteq (Y \cup \{x_{i_1}, \dots, x_{i_{a-1}}\}) \\ &\subseteq (Y \cup \{x_{i_1}, \dots, x_{i_{a-1}}, x_{i_{a+1}}, \dots, x_{i_b}\}). \end{aligned}$$

Applying the interval property (IG3) to the above sets and x_{i_a} gives that $Y \cup \{z_1, \dots, z_{a-1}, x_{i_a}\} = Y \cup \{x_{i_1}, \dots, x_{i_a}\} \in \mathcal{F}$. This contradicts the minimality of $z_a = x_{i_{a+1}}$. Therefore, no such b exists.

Therefore, $z_b \neq x_{i_a}$ for all $b > a$. Induction on l (as above) gives $z_l = x_{i_{l+1}}$ for all l such that $a < l < k$. Then $Y \cup \{x_{i_1}, \dots, \widehat{x_{i_a}}, \dots, x_{i_l}\} \in \mathcal{F}$ for all $1 \leq l \leq k$ and the proposition holds. \square

2.6. The Lattice of Flats.

2.6.1. Contractions. Let (E, \mathcal{F}) denote an interval greedoid and let $X \in \mathcal{F}$. Let \mathcal{F}/X denote the collection of subsets that can be added to X preserving feasibility:

$$\mathcal{F}/X = \{Y \subseteq E \setminus X : X \cup Y \in \mathcal{F}\}.$$

The pair $(\bigcup_{Y \in \mathcal{F}/X} Y, \mathcal{F}/X)$ is an interval greedoid, which we call the **contraction** of (E, \mathcal{F}) by X . Properties of contractions will be further developed in later sections. For now we record the following result, which is crucial to much of what follows.

Proposition 2.6.1. *Suppose (E, \mathcal{F}) is an interval greedoid and let $A \subseteq E$. Let U and V be maximal with respect to inclusion among the feasible sets contained in A . Then $\mathcal{F}/U = \mathcal{F}/V$.*

Proof. Let U and V be two maximal feasible subsets of A . Then $|U| = |V|$ (otherwise we can enlarge the smaller one using (IG2)). Suppose $W \in \mathcal{F}/U$ with $W \neq \emptyset$. Then $U \cup W \in \mathcal{F}$. Let $U = \{u_1 < \dots < u_r\}$ be a feasible ordering of U . Repeated application of (IG2) to U and $U \cup W$ gives a feasible ordering $\{u_1 < \dots < u_r < w_1 < \dots < w_s\}$ of $U \cup W$. Proposition 2.5.1 applied to $U \cup W$ and V gives an ordered subset $\{z_1 < \dots < z_t\}$ of $U \cup W$ such that $V \cup \{z_1, \dots, z_i\} \in \mathcal{F}$ for each $1 \leq i \leq t$, where $t = |U \cup W| - |V| = |W|$. If $z_1 \in U$, then $V \cup \{z_1\} \in \mathcal{F}$ and $V \cup \{z_1\} \subseteq A$, contradicting the maximality of V . Therefore, $z_1 \in W$ and the ordering of the z_i implies $z_i \in W$ for all $1 \leq i \leq t$. Since $t = |W|$, we have

$W = \{z_1, \dots, z_t\}$. Thus, $V \cup W \in \mathcal{F}$, or equivalently, $W \in \mathcal{F}/V$. Reversing the roles of U and V gives the reverse containment $\mathcal{F}/V \subseteq \mathcal{F}/U$. Thus, $\mathcal{F}/U = \mathcal{F}/V$. \square

Example 2.6.2 (Convex geometry on three colinear points). Consider the convex geometry on three colinear points from Example 2.3.2. The feasible sets are

$$\mathcal{F} = \left\{ \emptyset, \{x\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\} \right\}.$$

The following table shows \mathcal{F}/X for $X \in \mathcal{F}$.

X	\mathcal{F}/X
\emptyset	\mathcal{F}
$\{x\}$	$\{\emptyset, \{y\}, \{z\}, \{y, z\}\}$
$\{z\}$	$\{\emptyset, \{x\}, \{y\}, \{x, y\}\}$
$\{x, y\}$	$\{\emptyset, \{z\}\}$
$\{x, z\}$	$\{\emptyset, \{y\}\}$
$\{y, z\}$	$\{\emptyset, \{x\}\}$
$\{x, y, z\}$	$\{\emptyset\}$

From the table we notice that $\mathcal{F}/X = \mathcal{F}/Y$ implies $X = Y$. It turns out this is true for any antimatroid; see Example 2.6.10. \circ

2.6.2. Continuations. Let (E, \mathcal{F}) denote an interval greedoid and let $X \in \mathcal{F}$. The set of **continuations** $\Gamma(X)$ of X is the set of elements that can be added to X preserving feasibility:

$$\Gamma(X) = \{x \in E \setminus X : X \cup x \in \mathcal{F}\}.$$

Of course, if $X, Y \in \mathcal{F}$ and $\mathcal{F}/X = \mathcal{F}/Y$, then $\Gamma(X) = \Gamma(Y)$. The converse does not hold for arbitrary greedoids, but it does hold for interval greedoids.

Proposition 2.6.3. *Suppose (E, \mathcal{F}) is an interval greedoid. Then for all $X, Y \in \mathcal{F}$, we have $\Gamma(X) = \Gamma(Y)$ if and only if $\mathcal{F}/X = \mathcal{F}/Y$.*

Proof. Suppose $\Gamma(X) = \Gamma(Y)$. We argue that X is maximal among the feasible sets contained in $X \cup Y$. If not, then there exists $y \in Y$ such that $y \in \Gamma(X)$. Since $\Gamma(X) = \Gamma(Y)$, we have $y \in \Gamma(Y)$, contradicting that $Y \cap \Gamma(Y) = \emptyset$. Therefore, X is maximal among the feasible sets contained in $X \cup Y$. Similarly, Y is maximal among the feasible sets contained in $X \cup Y$. Therefore, $\mathcal{F}/X = \mathcal{F}/Y$ by Proposition 2.6.1. The reverse implication follows immediately from the definitions. \square

Example 2.6.4 (Vector Matroids). Let V be a vector space, and E a collection of vectors in V . \mathcal{F} consists of the linearly independent subsets of E . (See Example 2.2.1.) Let $X \in \mathcal{F}$. Then $\Gamma(X)$ consists of those vectors from E not in the span of X . \circ

Example 2.6.5 (Antimatroids). Let (E, τ) be a convex geometry and (E, \mathcal{F}) the corresponding antimatroid. If $X \in \mathcal{F}$, then $\Gamma(X) = \text{ext}(E \setminus X)$. \circ

Example 2.6.6 (Convex geometry on three colinear points). The following table shows that continuations of the feasible sets of the antimatroid in Example 2.3.2.

X	\emptyset	$\{x\}$	$\{z\}$	$\{x, y\}$	$\{x, z\}$	$\{y, z\}$	$\{x, y, z\}$	
$\Gamma(X)$	$\{x, z\}$	$\{y, z\}$	$\{x, y\}$	$\{z\}$	$\{y\}$	$\{x\}$	\emptyset	\circ

2.6.3. Flats. Let (E, \mathcal{F}) be an interval greedoid. Define an equivalence relation on \mathcal{F} by setting $X \sim Y$ if and only if $\mathcal{F}/X = \mathcal{F}/Y$. In light of Proposition 2.6.3, $X \sim Y$ if and only if $\Gamma(X) = \Gamma(Y)$. We write $[X]$ for the equivalence class of X :

$$[X] = \{Y \in \mathcal{F} : \mathcal{F}/X = \mathcal{F}/Y\} = \{Y \in \mathcal{F} : \Gamma(X) = \Gamma(Y)\}.$$

These equivalence classes are called the **flats** of (E, \mathcal{F}) .

The set Φ of flats of (E, \mathcal{F}) is a poset with partial order induced by reverse inclusion:

$$[X] \leq [Y] \text{ iff there exists } Z \in \mathcal{F}/Y \text{ such that } Y \cup Z \sim X.$$

In particular, if $Y \subseteq X$, then $[X] \leq [Y]$. (Note that some authors choose to use inclusion rather than reverse-inclusion to induce the partial order on Φ .)

The following result shows that Φ is a lower semimodular poset. In fact, Φ is a semimodular *lattice*; see Proposition 2.6.17.

Proposition 2.6.7. *Suppose (E, \mathcal{F}) is an interval greedoid. Let $X \in \mathcal{F}$ and suppose $X \cup x \in \mathcal{F}$ and $X \cup y \in \mathcal{F}$. If $[X \cup x] \neq [X \cup y]$, then $X \cup \{x, y\} \in \mathcal{F}$.*

Proof. Suppose (E, \mathcal{F}) is an interval greedoid and let $X \in \mathcal{F}$ with $X \cup x \in \mathcal{F}$ and $X \cup y \in \mathcal{F}$. If $X \cup \{x, y\} \notin \mathcal{F}$, then $X \cup x$ and $X \cup y$ are maximal among the feasible sets contained in $X \cup \{x, y\}$. Then Proposition 2.6.1 implies $X \cup x \sim X \cup y$. That is, $[X \cup x] = [X \cup y]$. \square

Example 2.6.8 (Vector Matroids). Let V be a vector space, E a collection of vectors from V , and \mathcal{F} the subsets of E that are linearly independent. For $X, Y \in \mathcal{F}$, $X \sim Y$ iff X and Y span the same subspace; and $[X] \leq [Y]$ iff the span of Y is contained in the span of X . \circ

Example 2.6.9 (Convex geometry on three colinear points). Consider the convex geometry on three colinear points (Example 2.3.2). The contractions of the corresponding antimatroid were described in Example 2.6.2. The poset of flats is illustrated in Figure 2. \circ

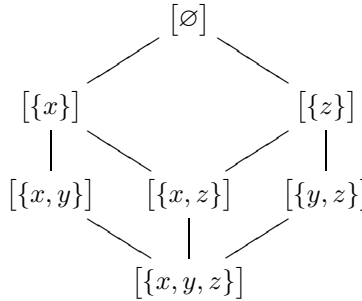


FIGURE 2. The poset of flats of the convex geometry on three colinear points.

Example 2.6.10 (Antimatroids). If (E, \mathcal{F}) is an antimatroid, then $[X] = \{X\}$ for all $X \in \mathcal{F}$. Indeed, if (E, τ) is the corresponding convex geometry and $\Gamma(X) = \Gamma(Y)$, then $E \setminus X = \tau(\text{ext}(E \setminus X)) = \tau(\text{ext}(E \setminus Y)) = E \setminus Y$ by Example 2.6.5. Thus, the poset of flats Φ of (E, \mathcal{F}) is isomorphic to \mathcal{F} ordered by reverse inclusion. \circ

Example 2.6.11 (Semimodular lattices). Let L be a finite lower semimodular lattice. Let E be the meet-irreducible elements of L , and let \mathcal{F} be the set system as in Proposition 2.4.1. The poset of flats of (E, \mathcal{F}) is naturally isomorphic to L .

Consider the following map $\phi : \mathcal{F} \rightarrow L$:

$$\phi(X) = \bigwedge_{e \in X} e.$$

It is constant on flats of (E, \mathcal{F}) , and therefore descends to a map from Φ to L , which is a poset isomorphism. See [BZ92b, Theorem 8.8.7]. \circ

2.6.4. Maps μ and ξ . Let (E, \mathcal{F}) be an interval greedoid and Φ its poset of flats. Define two maps $\mu : 2^E \rightarrow \Phi$ and $\xi : \Phi \rightarrow 2^E$ as follows.

- (μ) Define $\mu : 2^E \rightarrow \Phi$ on arbitrary subsets $A \subseteq E$ by $\mu(A) = [X]$, where X is maximal with respect to inclusion among the feasible sets contained in A .
- (ξ) Define $\xi : \Phi \rightarrow 2^E$ for $X \in \mathcal{F}$ by $\xi([X]) = \bigcup_{X' \sim X} X'$.

It follows from Proposition 2.6.1 that μ is well-defined. These maps are very important to what follows, and will be used to describe the meets and joins in Φ .

Proposition 2.6.12. *Suppose (E, \mathcal{F}) is an interval greedoid, Φ its lattice of flats and μ and ξ the maps defined above.*

- (1) $(\mu \circ \xi)(A) = A$ for all $A \in \Phi$. So ξ is injective.
- (2) $\mu : (2^E, \subseteq) \rightarrow (\Phi, \leq)$ is order-reversing.
- (3) $\xi : (\Phi, \leq) \rightarrow (2^E, \subseteq)$ is order-reversing.
- (4) $A \leq B$ if and only if $\xi(B) \subseteq \xi(A)$ for all $A, B \in \Phi$.
- (5) For all $Y \in \mathcal{F}$ and $A \in \Phi$, if $Y \subseteq \xi(A)$, then $A \leq [Y]$.

Proof. (1) Suppose that X is not maximal with respect to inclusion among the feasible sets contained in $\xi([X])$. Then there exists $x \in \xi([X]) - X$ such that $X \cup x \in \mathcal{F}$. Therefore, $x \in \mathcal{F}/X$ and $x \in X'$ for some $X' \sim X$ with $X' \neq X$. But $X' \sim X$ if and only if $\mathcal{F}/X = \mathcal{F}/X'$, so $x \in \mathcal{F}/X'$. This is a contradiction since $x \notin \mathcal{F}/X'$ if $x \in X'$. Thus, X is maximal, and so $\mu(\xi([X])) = [X]$.

(2) Suppose $A \subseteq B$. Let X be maximal with respect to inclusion among the feasible sets contained in A . Then there exists Y such that $X \subseteq Y \subseteq B$ and Y is maximal among the feasible sets contained in B . Therefore, $[Y] \leq [X]$. Hence, $\mu(B) \leq \mu(A)$.

(3) Suppose $[X] \leq [Y]$. If $e \in \xi([Y])$, then $e \in Y'$ for some $Y' \sim Y$. So $[X] \leq [Y] = [Y']$. Thus, there is a $Z \in \mathcal{F}/Y'$ such that $Y' \cup Z \sim X$. Therefore, $e \in Y' \subseteq (Y' \cup Z) \subseteq \xi([X])$.

(4) This follows from (1), (2) and (3).

(5) Suppose $Y \in \mathcal{F}$ and $Y \subseteq \xi([X])$. Then there exists Z containing Y that is maximal among the feasible sets contained in $\xi([X])$. Then $[Y] \geq [Z]$ since $Y \subseteq Z$ and $[Z] = [X]$ by Proposition 2.6.1. \square

Remark 2.6.13. Let (E, \mathcal{F}) be an interval greedoid. For $X \subset E$, the **rank** of X is the size of a maximal feasible set contained in X . X is closed if any proper superset of X has larger rank than X does. The **closure** of X is the smallest closed set containing X . (The uniqueness here follows from (IG2).)

If (E, \mathcal{F}) is a matroid without loops, then $(\xi \circ \mu)(A)$ is the closure of A (see Example 2.6.14 below). In general, though, all we can say is that $(\xi \circ \mu)(A)$ is contained in the closure of A . The containment follows from the fact that $(\mu \circ \xi \circ \mu)(A) = \mu(A)$ by Proposition 2.6.12(1). The fact that the containment is not necessarily an equality is shown in the following example.

Consider the convex geometry on the three colinear points x, y, z of Example 2.3.2. The empty set is feasible in the corresponding antimatroid and we have $\xi(\emptyset) = \emptyset$. But the closure of \emptyset is $\{y\}$ since the latter is not a feasible set (because $\{x, z\}$ is not a closed set in the convex geometry).

Example 2.6.14 (Matroids). Let (E, \mathcal{F}) be a matroid without loops. In this case $\xi(\Phi)$ consists exactly of the closed sets of the matroid.

Let $A \subset E$. As already remarked, Proposition 2.6.12(1) implies that $(\xi \circ \mu)(A)$ is contained in the closure of A . Conversely, suppose that e is in the closure of A . Since e is not a loop, $\{e\}$ is feasible, and can therefore be extended to a maximal feasible set X in $A \cup \{e\}$. Let Y be a maximal feasible set in A . Since e is in the closure of A , we have that Y is also a maximal feasible set inside $A \cup \{e\}$, and thus $X \sim Y$ by Proposition 2.6.1. It follows that $e \in Y \subset (\xi \circ \mu)(A)$. Thus $(\xi \circ \mu)(A)$ equals the closure of A . \circ

Example 2.6.15 (Antimatroids). Let (E, \mathcal{F}) be an antimatroid. Let X be feasible. Since $[X] = \{X\}$, $\xi([X]) = X$. Thus $\xi(\Phi)$ consists precisely of the feasible sets. \circ

Example 2.6.16 (Semimodular lattices). Let L be a lower semimodular lattice, and (E, \mathcal{F}) the associated interval greedoid. Let ϕ be the isomorphism from Φ to L , defined in Example 2.6.11. Let X be a feasible set. Then $\xi([X])$ consists of the set of meet-irreducibles f such that $f \geq \phi([X])$.

If f is in a feasible set $Y \sim X$, then $\phi([X]) = \phi([Y]) \leq f$, which proves one containment. For the other direction, let $f \geq \phi([X])$. Let Z be a feasible set with $\phi([Z]) = f$. Since f is meet-irreducible, $f \in Z$. Since $f \geq \phi([X])$, we know $[Z] \geq [X]$, which implies that $f \in Z \subset \xi([X])$, as desired. \circ

2.6.5. Lattice of flats. We have seen that Φ is a lower semimodular poset. It is also graded: the corank of any element $A \in \Phi$ is the size of any feasible set in A . The next result establishes that Φ is also a lattice.

Proposition 2.6.17. *If (E, \mathcal{F}) is an interval greedoid, then Φ is a lower semimodular lattice whose lattice operations are given by:*

$$A \vee B = \mu(\xi(A) \cap \xi(B)) \quad \text{and} \quad A \wedge B = \mu(\xi(A) \cup \xi(B))$$

for all $A, B \in \Phi$. That is, $A \vee B = [X]$, where X is maximal among the feasible sets contained in $\xi(A) \cap \xi(B)$, and $A \wedge B = [X]$, where X is maximal among the feasible sets contained in $\xi(A) \cup \xi(B)$.

Proof. By Proposition 2.6.7, Φ is a lower semimodular poset. It remains to show that Φ is a lattice. For $A, B \in \Phi$, define $j(A, B) = [X]$, where $X \in \mathcal{F}$ is maximal

among the feasible sets contained in $\xi(A) \cap \xi(B)$. Proposition 2.6.1 implies that $j(A, B)$ is well-defined. (Equivalently, $j(A, B) = \mu(\xi(A) \cap \xi(B))$.) Since $X \subseteq \xi(A)$, it follows from Proposition 2.6.12 that $A \leq [X] = j(A, B)$. Similarly, $B \leq j(A, B)$. Therefore, $j(A, B)$ is an upper bound of A and B .

It remains to show that $j(A, B)$ is the least upper bound. Suppose $A, B \leq [Y]$. Then $\xi([Y]) \subseteq \xi(A)$ and $\xi([Y]) \subseteq \xi(B)$ since ξ is order-reversing (Proposition 2.6.12). Therefore, $\xi([Y]) \subseteq \xi(A) \cap \xi(B)$. So there exists $X' \in \mathcal{F}$ such that $Y \subseteq X' \subseteq \xi(A) \cap \xi(B)$ and X' is maximal among the feasible sets contained in $\xi(A) \cap \xi(B)$. Therefore, by the maximality of X' and since $Y \subseteq X'$, we have $j(A, B) = [X'] \leq [Y]$.

For $A, B \in \Phi$, let $m(A, B) = [X]$, where X is maximal among the feasible sets contained in $\xi(A) \cup \xi(B)$. Let $A = [Y]$. Then $Y \subseteq \xi(A)$, so there exists $X' \supseteq Y$ such that X' is maximal among the feasible sets contained in $\xi(A) \cup \xi(B)$. Therefore, $m(A, B) = [X'] \leq [Y] = A$. Similarly, $m(A, B) \leq B$.

It remains to show that $m(A, B)$ is the greatest lower bound. Suppose $C \leq A$ and $C \leq B$. Then $\xi(A) \cup \xi(B) \subseteq \xi(C)$. So there exists a subset $X' \supseteq X$ that is maximal among the feasible sets contained in $\xi(C)$. Thus, $m(A, B) = [X] \geq [X'] = C$. \square

Example 2.6.18 (Antimatroids). Let (E, τ) be a convex geometry and (E, \mathcal{F}) the corresponding antimatroid. If $X, Y \in \mathcal{F}$, then $[X] \vee [Y] = [U]$, where U is maximal among the feasible sets contained in $X \cap Y$. By Example 2.6.10, U is unique and it follows that U is the complement of the closure of $(E \setminus X) \cup (E \setminus Y)$. Hence,

$$[X] \vee [Y] = \left[E \setminus \tau \left((E \setminus X) \cup (E \setminus Y) \right) \right]$$

for all $X, Y \in \mathcal{F}$. \circ

3. ORIENTED INTERVAL GREEDDOIDS

Throughout this section (E, \mathcal{F}) will denote an interval greedoid.

3.1. Signed flats. A **signed flat** of an interval greedoid (E, \mathcal{F}) is a pair $(A, \hat{\alpha})$ consisting of a flat A and a map $\hat{\alpha} : \Gamma(A) \rightarrow \{+, -\}$.

Define a partial order on signed flats as follows. If $(A, \hat{\alpha})$ and $(B, \hat{\beta})$ are signed flats of (E, \mathcal{F}) , let $(A, \hat{\alpha}) \leq (B, \hat{\beta})$ if $A \leq B$ (as flats in Φ) and if $\hat{\alpha}$ and $\hat{\beta}$ agree on $\Gamma(A) \cap \Gamma(B)$. (Reflexivity and anti-symmetry are straightforward to verify; transitivity follows by a simple application of (IG3).)

Define the product $(A, \hat{\alpha}) \circ (B, \hat{\beta})$ of two signed flats $(A, \hat{\alpha})$ and $(B, \hat{\beta})$ by

$$(A, \hat{\alpha}) \circ (B, \hat{\beta}) = (A \vee B, \hat{\alpha} \circ \hat{\beta}),$$

where, for $x \in \Gamma(A \vee B)$,

$$(\hat{\alpha} \circ \hat{\beta})(x) = \begin{cases} \hat{\alpha}(x), & \text{if } x \in \Gamma(A), \\ \hat{\beta}(x), & \text{otherwise.} \end{cases}$$

This product is well-defined because $\Gamma(A \vee B) \subseteq \Gamma(A) \cup \Gamma(B)$ (Proposition 3.1.3 below).

Example 3.1.1 (Antimatroids). Suppose (E, \mathcal{F}) is an antimatroid. As we saw in Example 2.6.5, the continuations of a feasible set X are the extreme points of the complement $E \setminus X$ in the convex geometry. Therefore, a signed flat $([X], \hat{\alpha})$ of the antimatroid is an assignment of $+$ or $-$ to each extreme point of $E \setminus X$. Figure 3 depicts a closed set C of a convex geometry; the extreme points of C are labelled by $+$ or $-$, the non-extreme points in C are labelled by 1, and the points in the exterior of C are labelled by 0.

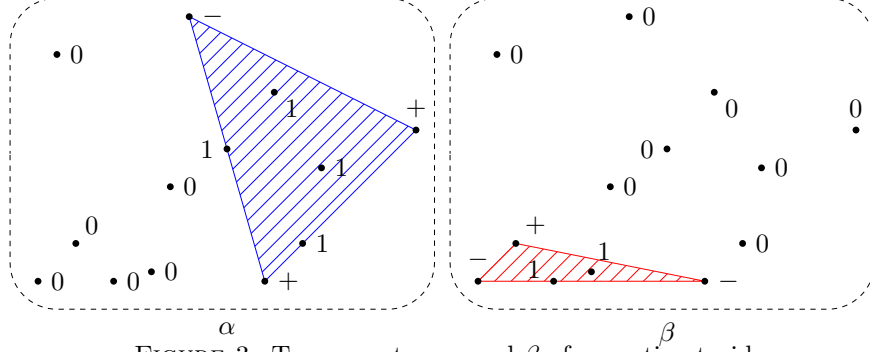


FIGURE 3. Two covectors α and β of an antimatroid.

The product of two signed flats $([X], \hat{\alpha})$ and $([Y], \hat{\beta})$ has a geometric interpretation. If $X' = E \setminus X$ and $Y' = E \setminus Y$, then form a new closed set Z' by taking the closure of $X' \cup Y'$; that is, $Z' = \tau(X' \cup Y')$. Note that the extreme points of Z' are contained in $\text{ext}(X') \cup \text{ext}(Y')$. The sign for each $z \in \text{ext}(Z')$ is $\hat{\alpha}(z)$ if $z \in \text{ext}(X')$, and $\hat{\beta}(z)$ otherwise. \circ

Example 3.1.2 (Matroids). Suppose (E, \mathcal{F}) is a matroid with no loops. If A is a flat of the matroid, then $\xi(A) = E \setminus \Gamma(A)$ is a closed set of the matroid. Therefore, a signed flat $(A, \hat{\alpha})$ is an assignment of a sign $+$ or $-$ to each element of the complement of the closed set A . If we extend this by assigning 0 to each element of A , then $\hat{\alpha}$ induces a *covector* in the sense of oriented matroids. (See §3.4.1.) \circ

Among other things, the following establishes that the product of signed flats is well-defined.

Proposition 3.1.3. *Let (E, \mathcal{F}) be an interval greedoid, $A, B \in \Phi$ and $X \in \mathcal{F}$.*

- (1) *If $B \leq [X]$ and $x \in \Gamma(X)$, then either $x \in \Gamma(B)$ or $B \leq [X \cup x]$.*
- (2) *If $A \leq B$, then $\Gamma(B) \subseteq \Gamma(A) \cup \xi(A)$.*
- (3) *$\Gamma(A \vee B) \subseteq \Gamma(A) \cup \Gamma(B)$.*
- (4) *$\Gamma(A \vee B) \cup \xi(A \vee B) \subseteq (\Gamma(A) \cup \xi(A)) \cap (\Gamma(B) \cup \xi(B))$.*

Proof. (1) Pick $Y \in \mathcal{F}$ such that $B = [Y]$. Suppose $[Y] \leq [X]$ and let $x \in \Gamma(X)$. Then there exists $Z \in \mathcal{F}/X$ such that $X \cup Z \sim Y$. Applying axiom (IG2) repeatedly to X and $X \cup Z$ yields a sequence $X \subset (X \cup z_1) \subset (X \cup \{z_1, z_2\}) \subset \dots \subset (X \cup Z)$ of feasible sets. Put $X_0 = X$ and let $X_i = X_{i-1} \cup z_i$ for $1 \leq i \leq r = |Z|$.

$$\begin{array}{c}
 X \cup \{z_1\} \hookrightarrow X \cup \{z_1, z_2\} \hookrightarrow \dots \hookrightarrow X \cup Z \sim Y \\
 \swarrow \quad \searrow \\
 X \hookrightarrow X \cup \{x\} \hookrightarrow X \cup \{z_1, x\} \hookrightarrow \dots \hookrightarrow X \cup Z \cup \{x\}
 \end{array}$$

Since $x \in \Gamma(X)$, $X \cup x \in \mathcal{F}$. If $[X \cup x] = [X \cup z_1]$, then $[Y] = [X \cup Z] \leq [X \cup z_1] = [X \cup x]$ since $X \subseteq X \cup Z$. If $[X \cup x] \neq [X \cup z_1]$, then $X \cup \{x, z_1\} \in \mathcal{F}$ by Proposition 2.6.7 since $x, z_1 \in \Gamma(X)$. Thus, $x, z_2 \in \Gamma(X \cup z_1)$. If $[X \cup \{z_1, x\}] = [X \cup \{z_1, z_2\}]$, then $[Y] \leq [X \cup x]$ using a similar argument as in the previous case. If $[X \cup \{z_1, x\}] \neq [X \cup \{z_1, z_2\}]$, then $X \cup \{z_1, z_2, x\} \in \mathcal{F}$ by Proposition 2.6.7. Continuing in this manner we get either that $[Y] \leq [X \cup x]$ or $(X \cup Z) \cup x \in \mathcal{F}$. That is, either $B = [Y] \leq [X \cup x]$, or $x \in \Gamma(Y) = \Gamma(B)$. This proves the statement.

(2) Pick $X, Y \in \mathcal{F}$ such that $A = [Y]$ and $B = [X]$. If $[Y] \leq [X]$ and $x \in \Gamma(Y)$, then $x \in \Gamma(Y)$ or $[Y] \leq [X \cup x]$ by (1). In the latter case, $x \in \xi(X \cup x) \subseteq \xi(Y)$ since ξ is order-reversing. Thus, $x \in \Gamma(Y)$ or $x \in \xi(Y)$.

(3) Pick $X \in \mathcal{F}$ such that $A \vee B = [X]$. Let $x \in \Gamma(X)$. Then $X \cup x \in \mathcal{F}$ and $[X \cup x] < [X]$. Since $[X] = A \vee B$, it follows that $[X \cup x]$ is not above both A and B . If $A \not\leq [X \cup x]$, then the above applied to $A \vee B$ and A gives that $x \in \Gamma(A)$ since $A \not\leq [X \cup x]$. Similarly, if $B \not\leq [X \cup x]$, then $x \in \Gamma(B)$. Hence, $x \in \Gamma(A) \cup \Gamma(B)$.

(4) Since ξ is order-reversing, it follows that $\xi(A \vee B) \subseteq \xi(A) \cap \xi(B)$. (4) now follows from (2). \square

The next result collects some properties of the product and partial order of signed flats.

Proposition 3.1.4. *Let $(A, \hat{\alpha})$ and $(B, \hat{\beta})$ denote two signed flats over an interval greedoid (E, \mathcal{F}) . Then*

- (1) $(A, \hat{\alpha}) \leq (B, \hat{\beta})$ if and only if $(A, \hat{\alpha}) \circ (B, \hat{\beta}) = (B, \hat{\beta})$.
- (2) $(A, \hat{\alpha}) \leq (A, \hat{\alpha}) \circ (B, \hat{\beta})$.
- (3) If $A \leq B$, then $(B, \hat{\beta}) \circ (A, \hat{\alpha}) = (B, \hat{\beta})$.
- (4) The product \circ is associative.
- (5) $(A, \hat{\alpha}) \circ (B, \hat{\beta}) \circ (A, \hat{\alpha}) = (A, \hat{\alpha}) \circ (B, \hat{\beta})$.
- (6) $(A, \hat{\alpha}) \circ (A, \hat{\alpha}) = (A, \hat{\alpha})$.

Proof. (1) Suppose $(A, \hat{\alpha}) \circ (B, \hat{\beta}) = (B, \hat{\beta})$. Since $(A, \hat{\alpha}) \circ (B, \hat{\beta}) = (A \vee B, \hat{\alpha} \circ \hat{\beta})$, it follows that $B = A \vee B$ and that $\hat{\beta} = \hat{\alpha} \circ \hat{\beta}$. Therefore, $A \leq B$ and $\hat{\beta}(x) = (\hat{\alpha} \circ \hat{\beta})(x) = \hat{\alpha}(x)$ for all $x \in \Gamma(A) \cap \Gamma(B)$. Thus, $(A, \hat{\alpha}) \leq (B, \hat{\beta})$.

Conversely, suppose $(A, \hat{\alpha}) \leq (B, \hat{\beta})$. Then $A \leq B$ and $\hat{\alpha}(x) = \hat{\beta}(x)$ for all $x \in \Gamma(A) \cap \Gamma(B)$. Therefore, $A \vee B = B$. It remains to show that $(\hat{\alpha} \circ \hat{\beta})(x) = \hat{\beta}(x)$ for all $x \in \Gamma(A \vee B) = \Gamma(B)$. Let $x \in \Gamma(B)$. If $x \in \Gamma(A)$, then $(\hat{\alpha} \circ \hat{\beta})(x) = \hat{\alpha}(x)$ and $\hat{\alpha}(x) = \hat{\beta}(x)$ since $\hat{\alpha}$ and $\hat{\beta}$ agree on $\Gamma(A) \cap \Gamma(B)$. If $x \notin \Gamma(A)$, then $(\hat{\alpha} \circ \hat{\beta})(x) = \hat{\beta}(x)$. Therefore, $\hat{\beta}$ and $\hat{\alpha} \circ \hat{\beta}$ agree on $\Gamma(A \vee B)$.

(2) First note that $A \leq A \vee B$ by the definition of \vee . We need only show that $\hat{\alpha}$ and $\hat{\alpha} \circ \hat{\beta}$ agree on $\Gamma(A) \cap \Gamma(A \vee B)$, which follows from the definition of $\hat{\alpha} \circ \hat{\beta}$. Therefore, $(A, \hat{\alpha}) \leq (A \vee B, \hat{\alpha} \circ \hat{\beta}) = (A, \hat{\alpha}) \circ (B, \hat{\beta})$.

(3) If $A \leq B$, then $A \vee B = B$, so $\Gamma(A \vee B) = \Gamma(B)$. So the domains of $\hat{\beta} \circ \hat{\alpha}$ and $\hat{\beta}$ are the same. And from the definition of \circ , if $x \in \Gamma(B)$, then $(\hat{\beta} \circ \hat{\alpha})(x) = \hat{\beta}(x)$.

(4), (5) and (6) are straightforward to verify using similar arguments. \square

3.2. Covectors. Let $(A, \hat{\alpha})$ denote a signed flat. Then $\hat{\alpha} : \Gamma(A) \rightarrow \{+, -\}$ can be extended to a map $\alpha : E \rightarrow \{0, +, -, 1\}$ as follows,

$$\alpha(e) = \begin{cases} \hat{\alpha}(e), & \text{if } e \in \Gamma(A), \\ 0, & \text{if } e \in \xi(A), \\ 1, & \text{otherwise.} \end{cases}$$

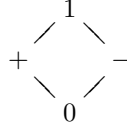
This map α is called the **covector** of the signed flat $(A, \hat{\alpha})$.

Example 3.2.1 (Antimatroids). Let E be a finite subset of \mathbb{R}^n and $\tau(X) = \text{conv}(X) \cap E$. Let (E, \mathcal{F}) denote the corresponding upper interval greedoid. Suppose $(A, \hat{\alpha})$ is a signed flat of (E, \mathcal{F}) and let $X \in \mathcal{F}$ with $A = [X]$. Then the covector α of the signed flat is obtained by assigning 0 to the points in the exterior of $E \setminus X$, $\hat{\alpha}(x)$ to the points $e \in \text{ext}(E \setminus X)$, and 1 to the non-extreme points contained in $E \setminus X$. See Figure 3 for an example. \circ

Note that a signed flat $(A, \hat{\alpha})$ can be recovered from its covector α . Indeed, the set of indices $e \in E$ such that $\alpha(e) = 0$ is precisely the set $\xi(A)$, from which A can be recovered (Proposition 2.6.12). Therefore, there exists a map from the set of covectors of (E, \mathcal{F}) to the lattice of flats Φ ,

$$\text{supp}(\alpha) = \mu(\{x \in E : \alpha(x) = 0\}).$$

The product on signed flats can be formulated for covectors as follows. Let $\alpha, \beta : E \rightarrow \{0, +, -, 1\}$ be the covectors of the signed flats $(A, \hat{\alpha})$, $(B, \hat{\beta})$, respectively. Define a partial order on the symbols $0, +, -, 1$ according to the following Hasse diagram.



Define

$$(\alpha \star \beta)(e) = \begin{cases} \beta(e), & \text{if } \beta(e) > \alpha(e), \\ \alpha(e), & \text{otherwise} \end{cases}$$

and

$$(\alpha \circ \beta)(e) = \begin{cases} (\alpha \star \beta)(e), & \text{if } e \in \Gamma(A \vee B) \cup \xi(A \vee B), \\ 1, & \text{otherwise.} \end{cases}$$

Example 3.2.2 (Covector multiplication in an antimatroid). Let α and β be covectors of (E, \mathcal{F}) from Example 3.2.1 and X' and Y' their underlying closed sets. The covector $\alpha \circ \beta$ is obtained as follows. Let $Z' = \tau(X' \cup Y')$. Then $(\alpha \circ \beta)(z)$ is 0 if z is in the exterior of Z' , 1 if z is a non-extreme point of Z' , $\alpha(z)$ if $z \in \text{ext}(X')$, and $\beta(z)$ otherwise. Figure 4 depicts the product of the covectors from Figure 3. \circ

Proposition 3.2.3. Suppose α and β are the covectors of the signed flats $(A, \hat{\alpha})$ and $(B, \hat{\beta})$, respectively. Then the covector γ of $(A, \hat{\alpha}) \circ (B, \hat{\beta})$ is $\alpha \circ \beta$.

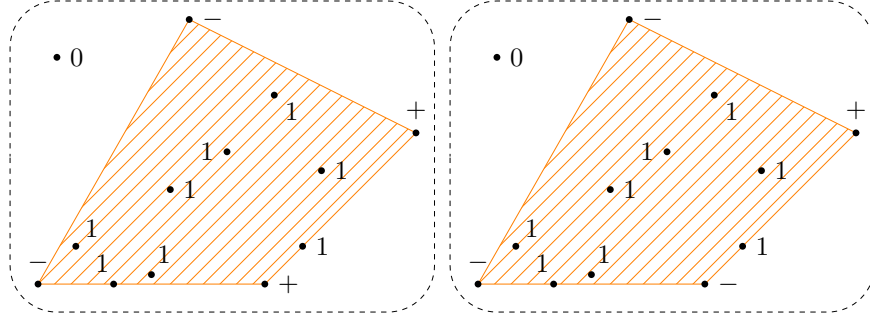


FIGURE 4. The products $\alpha \circ \beta$ (left) and $\beta \circ \alpha$ (right) of the covectors α and β in Figure 3.

Proof. By definition, the covector γ of the signed flat $(A \vee B, \hat{\alpha} \circ \hat{\beta})$ is given by: $\gamma(e) = (\hat{\alpha} \circ \hat{\beta})(e)$ if $e \in \Gamma(A \vee B)$; $\gamma(e) = 0$ if $e \in \xi(A \vee B)$; and $\gamma(e) = 1$ otherwise.

Suppose $e \notin \Gamma(A \vee B) \cup \xi(A \vee B)$. By the definition of the product of covectors, $(\alpha \circ \beta)(e) = 1$. Hence, $(\alpha \circ \beta)(e) = \gamma(e)$.

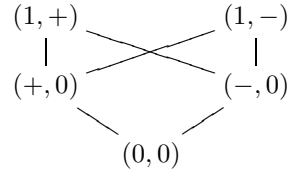
Suppose $e \in \xi(A \vee B)$. Then $e \in \xi(A)$ and $e \in \xi(B)$ since ξ is order-reversing. This implies that $\alpha(e) = \beta(e) = 0$, hence $(\alpha \circ \beta)(e) = 0$. Therefore, $(\alpha \circ \beta)(e) = \gamma(e)$.

Suppose $e \in \Gamma(A \vee B)$. By Proposition 3.1.3(3), $e \in \Gamma(A) \cup \Gamma(B)$. If $e \in \Gamma(A)$, then $\beta(e) \not> \alpha(e)$, so $(\alpha \circ \beta)(e) = \alpha(e) = \hat{\alpha}(e)$. If $e \notin \Gamma(A)$, then $e \in \Gamma(B) \cap \xi(A)$ and so $\beta(e) > 0 = \alpha(e)$. Hence, $(\alpha \circ \beta)(e) = \beta(e) = \hat{\beta}(e)$. Therefore, $(\alpha \circ \beta)(e) = (\hat{\alpha} \circ \hat{\beta})(e)$ for all $e \in \Gamma(A \vee B)$. \square

Example 3.2.4. Let $E = \{x, y\}$ and $\mathcal{F} = \{\emptyset, \{y\}, \{x, y\}\}$. Then (E, \mathcal{F}) is an upper interval greedoid. There are five covectors of (E, \mathcal{F}) , described in the following table.

$[X]$	$\Gamma(X)$	covectors over $[X]$
$[\emptyset]$	$\{y\}$	$(1, +), (1, -)$
$[\{y\}]$	$\{x\}$	$(+, 0), (-, 0)$
$[\{x, y\}]$	\emptyset	$(0, 0)$

The partial order on these covectors is illustrated below.



Observe that the product of two covectors α and β can be computed using \star , or using the following identity:

$$\alpha \circ \beta = \begin{cases} \beta, & \text{if } \beta > \alpha, \\ \alpha, & \text{otherwise.} \end{cases}$$

For example, $(+, 0) \circ (-, 0) = (+, 0)$ and $(+, 0) \circ (1, -) = (1, -)$. \circ

Let α and β be covectors of (E, \mathcal{F}) . The **separation set** of α and β is

$$S(\alpha, \beta) = \{e \in E : \alpha(e) = -\beta(e) \in \{+, -\}\}.$$

Note that $S(\alpha, \beta) \subseteq \Gamma(\text{supp}(\alpha)) \cap \Gamma(\text{supp}(\beta))$.

The next result establishes some properties about covectors. See also Proposition 3.1.4.

Lemma 3.2.5. *Let α and β be covectors of an interval greedoid (E, \mathcal{F}) .*

- (1) $\alpha \leq \beta$ if and only if $\alpha \circ \beta = \beta$.
- (2) $\alpha \leq \beta$ if and only if $\alpha(e) \leq \beta(e)$ for all $e \in E$.
- (3) $\alpha \leq \beta$ if and only if $S(\alpha, \beta) = \emptyset$ and $\text{supp}(\alpha) \leq \text{supp}(\beta)$.
- (4) If $\alpha(e) = 1$ or $\beta(e) = 1$, then $(\alpha \circ \beta)(e) = 1 = (\beta \circ \alpha)(e)$.

Proof. Let $A = \text{supp}(\alpha)$ and $B = \text{supp}(\beta)$. By definition, $\alpha \leq \beta$ if and only if $A \leq B$ and α and β agree on $\Gamma(A) \cap \Gamma(B)$.

(1) This follows from Proposition 3.1.4 and Proposition 3.2.3.

(2) Suppose $\alpha \leq \beta$ and let $e \in E$. If $e \notin \xi(B) \cup \Gamma(B)$, then $\beta(e) = 1$, so $\alpha(e) \leq \beta(e)$. If $e \in \xi(A)$, then $\alpha(e) = 0$, so $\alpha(e) \leq \beta(e)$. So suppose $e \in \xi(B) \cup \Gamma(B)$ and $e \notin \xi(A)$. Then $e \in \Gamma(A) \cap \Gamma(B)$, by Proposition 3.1.3. Then $\alpha(e) \leq \beta(e)$ because α and β agree on $\Gamma(A) \cap \Gamma(B)$.

Conversely, suppose $\alpha(e) \leq \beta(e)$ for all $e \in E$. Since $\{e : \beta(e) = 0\} \subseteq \{e : \alpha(e) = 0\}$, we have $\text{supp}(\alpha) \leq \text{supp}(\beta)$. If $e \in \Gamma(A) \cap \Gamma(B)$, then $\alpha(e), \beta(e) \in \{+, -\}$, which implies $\alpha(e) = \beta(e)$ because $\alpha(e) \leq \beta(e)$. Thus, $\alpha \leq \beta$.

(3) If $\alpha \leq \beta$, then $A \leq B$, and $S(\alpha, \beta) = \emptyset$ because $S(\alpha, \beta) \subseteq \Gamma(A) \cap \Gamma(B)$. Conversely, if $A \leq B$ and $S(\alpha, \beta) = \emptyset$, then $\alpha(e) = \beta(e)$ for all $e \in \Gamma(A) \cap \Gamma(B)$, so $\alpha \leq \beta$.

(4) If $(\alpha \circ \beta)(e) \neq 1$, then $e \in \Gamma(A \vee B) \cup \xi(A \vee B) \subseteq (\Gamma(A) \cup \xi(A)) \cap (\Gamma(B) \cup \xi(B))$ by Proposition 3.1.3(4). Hence, $\alpha(e) \neq 1$ and $\beta(e) \neq 1$. \square

Remark 3.2.6. The converse of (4) is false. Counter-examples are depicted in Figure 5. They also illustrate that the following containments can be proper.

$$\Gamma(A \vee B) \cup \xi(A \vee B) \subseteq (\Gamma(A) \cup \xi(A)) \cap ((\Gamma(B) \cup \xi(B))),$$

$$\xi(A' \vee B') \subseteq \xi(A') \cap \xi(B').$$

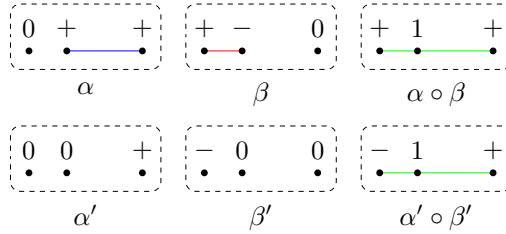


FIGURE 5. Counter-examples to the converse of Lemma 3.2.5 (4).

3.3. Oriented interval greedoids. For any covector α , let $-\alpha$ be the covector obtained from α by replacing $+$ with $-$ and $-$ with $+$.

Definition 3.3.1. An *oriented interval greedoid* is a triple $(E, \mathcal{F}, \mathcal{G})$, where (E, \mathcal{F}) is an interval greedoid and \mathcal{G} is a set of covectors of (E, \mathcal{F}) satisfying the following axioms.

- (OG1) The map $\text{supp} : \mathcal{G} \rightarrow \Phi$ is surjective.
- (OG2) If $\alpha \in \mathcal{G}$, then $-\alpha \in \mathcal{G}$.
- (OG3) If $\alpha, \beta \in \mathcal{G}$, then $\alpha \circ \beta \in \mathcal{G}$.
- (OG4) If $\alpha, \beta \in \mathcal{G}$, $x \in S(\alpha, \beta)$ and $(\alpha \circ \beta)(x) \neq 1$, then there exists $\gamma \in \mathcal{G}$ such that $\gamma(x) = 0$ and for all $y \notin S(\alpha, \beta)$, if $(\alpha \circ \beta)(y) \neq 1$, then $\gamma(y) = (\alpha \circ \beta)(y) = (\beta \circ \alpha)(y)$.

As we will see in Section §3.4.1, these conditions are modelled on the covector axioms for oriented matroids. In the next section we will present various examples of oriented interval greedoids. We record here the following observation.

Lemma 3.3.2. *Suppose α and β are covectors of an oriented interval greedoid. If $(\alpha \circ \beta)(y) \neq (\beta \circ \alpha)(y)$, then $\alpha(y) = -\beta(y) \in \{+, -\}$ (that is, $y \in S(\alpha, \beta)$.)*

Proof. Let $C = \text{supp}(\alpha \circ \beta) = \text{supp}(\beta \circ \alpha)$. Then $(\alpha \circ \beta)(y) = 1$ iff $y \notin \Gamma(C) \cup \xi(C)$ iff $(\beta \circ \alpha)(y) = 1$. Similarly, $(\alpha \circ \beta)(y) = 0$ iff $y \in \xi(C)$ iff $(\beta \circ \alpha)(y) = 0$. Thus $\alpha(y), \beta(y) \in \{+, -\}$. The result follows. \square

Corollary 3.3.3. *Suppose α and β are covectors of an oriented interval greedoid. Then $(\alpha \circ \beta)(y) = (\beta \circ \alpha)(y)$ for all $y \notin S(\alpha, \beta)$.*

3.4. Examples. This section presents some examples of oriented interval greedoids.

3.4.1. Oriented Matroids. Let E be a finite set. An **oriented matroid** is a collection \mathcal{O} of maps from E to $\{0, +, -\}$ that satisfies the following axioms.

- (OM1) \mathcal{O} contains the map $z(e) = 0$ for all $e \in E$.
- (OM2) If $\alpha \in \mathcal{O}$, then $-\alpha \in \mathcal{O}$.
- (OM3) If $\alpha, \beta \in \mathcal{O}$, then $\alpha \circ \beta \in \mathcal{O}$, where

$$(\alpha \circ \beta)(e) = \begin{cases} \alpha(e), & \text{if } \alpha(e) \neq 0, \\ \beta(e), & \text{if } \alpha(e) = 0. \end{cases}$$

- (OM4) Suppose $\alpha, \beta \in \mathcal{O}$ and let $S(\alpha, \beta) = \{e \in E : \alpha(e) = -\beta(e) \neq 0\}$. For every $e \in S(\alpha, \beta)$ there exists $\gamma \in \mathcal{O}$ with $\gamma(e) = 0$ and $\gamma(f) = (\alpha \circ \beta)(f) = (\beta \circ \alpha)(f)$ for all $f \notin S(\alpha, \beta)$.

If \mathcal{O} is an oriented matroid, then the set of zeros of the elements of \mathcal{O} form the closed sets of a matroid (E, \mathcal{F}) . The matroid (E, \mathcal{F}) is the **underlying matroid** of the oriented matroid and \mathcal{O} is said to be an oriented matroid on (E, \mathcal{F}) .

Theorem 3.4.1. *Suppose (E, \mathcal{F}) is a matroid without loops. Then \mathcal{O} is an oriented matroid with underlying matroid (E, \mathcal{F}) if and only if $(E, \mathcal{F}, \mathcal{O})$ is an oriented interval greedoid.*

Proof. Let (E, \mathcal{F}) be a matroid and let $(E, \mathcal{F}, \mathcal{G})$ be an oriented interval greedoid. Since $\xi(A) = E \setminus \Gamma(A)$ for any flat A of a matroid without loops, a covector of (E, \mathcal{F}) takes values in $\{0, +, -\}$. Therefore, \mathcal{G} is a collection of maps from E to $\{0, +, -\}$, and \mathcal{G} satisfies (OM1)–(OM4) since it satisfies (OG1)–(OG4). So \mathcal{G} is an oriented matroid.

Conversely, suppose that \mathcal{O} is an oriented matroid with underlying matroid (E, \mathcal{F}) . If $\alpha \in \mathcal{O}$, then the set $\zeta(\alpha)$ of zeros of α is a closed set of the matroid, and there is a unique flat A satisfying $\xi(A) = \zeta(\alpha)$. Therefore, α gives a signed flat $(A, \alpha|_{\Gamma(A)})$, and the covector of this signed flat is α . So \mathcal{O} is a set of covectors of the interval greedoid (E, \mathcal{F}) . It is straightforward to check that the axioms for an oriented interval greedoid are satisfied by \mathcal{O} . \square

3.4.2. Antimatroids. Next we show that the set of all covectors of an antimatroid forms an oriented interval greedoid. This collection of covectors, viewed as a poset, is the central object of study in the work of Billera, Hsiao, and Provan [BHP08]. We also show that this is the only oriented interval greedoid structure on an antimatroid.

We begin with an example to illustrate how to obtain a covector γ satisfying (OG4).

Example 3.4.2. Let α and β be the covectors in Figure 3. Then $S(\alpha, \beta) = \{x\}$, where x is the vertex that is circled in Figure 6. Let γ be the covector in Figure 6. Then $\gamma(x) = 0$; and for all $y \notin S(\alpha, \beta)$:

- (1) if $(\alpha \circ \beta)(y) = 0$, then $\gamma(y) = (\beta \circ \alpha)(y) = 0$.
- (2) if $(\alpha \circ \beta)(y) = +$, then $\gamma(y) = (\beta \circ \alpha)(y) = +$.
- (3) if $(\alpha \circ \beta)(y) = -$, then $\gamma(y) = (\beta \circ \alpha)(y) = -$.
- (4) if $(\alpha \circ \beta)(y) = 1$, then $\gamma(y) \neq 0$.

○

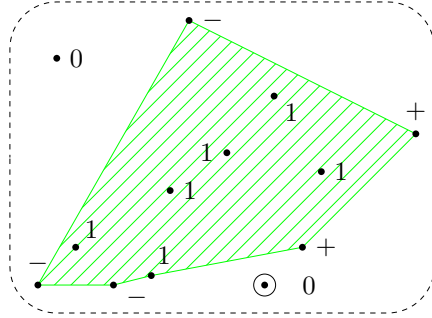


FIGURE 6. If α and β are the two covectors in Figure 3 and x is the circled vertex, then the covector γ illustrated here satisfies the conditions of (OG4).

Theorem 3.4.3. Suppose (E, \mathcal{F}) is an upper interval greedoid. Let \mathcal{G} denote the set of all covectors of (E, \mathcal{F}) . Then $(E, \mathcal{F}, \mathcal{G})$ is an oriented interval greedoid.

Proof. We will show that (OG1)–(OG4) hold.

(OG1) Suppose $A \in \Phi$ is a flat. Let $(A, \hat{\alpha})$ be a signed flat of (E, \mathcal{F}) and let α denote the covector of this signed flat. Then $\alpha \in \mathcal{G}$ and $\text{supp}(\alpha) = A$.

(OG2) Suppose $\alpha \in \mathcal{G}$. Let $A = \text{supp}(\alpha)$. Then $(A, -\alpha|_{\Gamma(A)})$ is a signed flat of (E, \mathcal{F}) . The covector of this signed flat is precisely $-\alpha$. So $-\alpha \in \mathcal{G}$.

(OG3) If $\alpha, \beta \in \mathcal{G}$, then $\alpha \circ \beta$ is a covector of (E, \mathcal{F}) , so $\alpha \circ \beta \in \mathcal{G}$.

(OG4) Suppose $\alpha, \beta \in \mathcal{G}$ and $x \in S(\alpha, \beta)$ satisfies $(\alpha \circ \beta)(x) \neq 1$. Let (E, τ) denote the convex geometry that is complementary to (E, \mathcal{F}) (see §2.3.2). Let

$A = \text{supp}(\alpha)$ and $B = \text{supp}(\beta)$. Then $A = [X]$ and $B = [Y]$ for some $X, Y \in \mathcal{F}$. Let $X' = E \setminus X$ and $Y' = E \setminus Y$. Then X' and Y' are closed sets in (E, τ) .

Step 1: We show that x is an extreme point of $\tau(X' \cup Y')$. Since $(\alpha \circ \beta)(x) \notin \{0, 1\}$, we have $x \in \Gamma(A \vee B)$. Since $A \vee B = [E \setminus \tau(X' \cup Y')]$ (Example 2.6.18), we have $\Gamma(A \vee B) = \text{ext}(\tau(X' \cup Y'))$ by Example 2.6.5. Thus, $x \in \text{ext}(\tau(X' \cup Y'))$.

Step 2: We define γ . Let $Z' = \tau(X' \cup Y') - x$. Since x is an extreme point of $\tau(X' \cup Y')$, it follows that x is not in Z' . Hence Z' is a closed set not containing x . Let $Z = E \setminus Z'$. Then $Z \in \mathcal{F}$ and $x \in Z$. Define a map $\gamma' : \Gamma(Z) \rightarrow \{+, -\}$ for $y \in \Gamma(Z)$ as follows: if $y \in \Gamma(A \vee B)$, then set $\gamma'(y) = (\alpha \circ \beta)(y)$; otherwise arbitrarily set $\gamma'(y)$ to be $+$ or $-$. Let γ be the covector of the signed flat $([Z], \gamma')$.

Step 3: γ has the desired properties. First note that $\gamma \in \mathcal{G}$ since γ is a covector of (E, \mathcal{F}) . Next observe that $\gamma(x) = 0$ since $x \in Z \subseteq \xi([Z])$. Let $y \notin S(\alpha, \beta)$.

Suppose that $(\alpha \circ \beta)(y) \neq 1$. Then $y \in \Gamma(A \vee B) \cup \xi(A \vee B)$. Since $\xi(A \vee B) = E \setminus \tau(X' \cup Y') \subseteq Z \subseteq \xi([Z])$, if $y \in \xi(A \vee B)$, then $\gamma(y) = 0 = (\alpha \circ \beta)(y) = (\beta \circ \alpha)(y)$. On the other hand, if $y \in \Gamma(A \vee B) = \text{ext}(\tau(X' \cup Y'))$, then y is an extreme point of $Z' = \tau(X' \cup Y') - x$ (since $y \neq x$). Equivalently, $y \in \Gamma(Z)$. So, by definition of γ and because $y \notin S(\alpha, \beta)$, $\gamma(y) = (\alpha \circ \beta)(y) = (\beta \circ \alpha)(y)$. \square

Proposition 3.4.4. *Let (E, \mathcal{F}) be an antimatroid. Then the only oriented structure on (E, \mathcal{F}) is that constructed in Theorem 3.4.3.*

Proof. Let $(E, \mathcal{F}, \mathcal{G})$ be an oriented interval greedoid. Let α be an arbitrary covector. We wish to show that $\alpha \in \mathcal{G}$.

Let $\Gamma(\text{supp}(\alpha)) = X = \{x_1, \dots, x_r\}$. Let $Y = E \setminus (X \cup \text{supp}(\alpha))$. For any $x \in X$, $Y_x = \text{supp}(\alpha) \cup (X \setminus \{x\})$ is feasible. Also, $\Gamma(Y_x) \cap \Gamma(\text{supp}(\alpha)) = \{x\}$. By (OIG1), we can find a covector $\beta_x \in \mathcal{G}$ with $\text{supp}(\beta_x) = Y_x$. By (OIG2), we can choose β_x so that β_x agrees with α on x . Now $\beta_{x_1} \circ \dots \circ \beta_{x_r} = \alpha$ is in \mathcal{G} . \square

3.4.3. Complexified Hyperplane Arrangements. An (essential) **real hyperplane arrangement** is a finite set of hyperplanes $\{\Theta_1, \Theta_2, \dots, \Theta_n\}$ in \mathbb{R}^d satisfying $\bigcap \Theta_i = \{\vec{0}\}$. Let $E = \{1, 2, \dots, n\}$ and for each $e \in E$ fix a linear form $\ell_e : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $\Theta_e = \ker(\ell_e)$. Extending scalars, we can also think of ℓ_e as defining a linear map from \mathbb{C}^d to \mathbb{C} . Define H_e to be the kernel of this map. It is a hyperplane in \mathbb{C}^d . The collection $\mathcal{A} = \{H_1, \dots, H_n\}$ forms a **complexified hyperplane arrangement**. Also define $H_e^{\Re} = \{\vec{z} \in \mathbb{C}^d : \Im(\ell_e(\vec{z})) = 0\}$.

(Note that not all complex hyperplane arrangements are complexified arrangements; that is to say, not all complex hyperplane arrangements arise from a real hyperplane arrangement in the way we have just described.)

For any $z = x + iy \in \mathbb{C}$, let

$$\sigma_{\Re}(x + iy) = \begin{cases} 1, & \text{if } y \neq 0, \\ +, & \text{if } y = 0, x > 0, \\ -, & \text{if } y = 0, x < 0, \\ 0, & \text{if } y = 0, x = 0, \end{cases} \quad \sigma_{\Im}(x + iy) = \begin{cases} +, & \text{if } y > 0, \\ -, & \text{if } y < 0, \\ 0, & \text{if } y = 0, \end{cases}$$

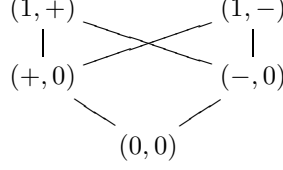


FIGURE 7. The poset of covectors of the complex hyperplane arrangement in \mathbb{C} .

and for every $\vec{z} \in \mathbb{C}^d$, let

$$\alpha_{\vec{z}}(h) = \begin{cases} \sigma_{\Im}(\ell_i(\vec{z})), & \text{if } h = H_i^{\Re}, \\ \sigma_{\Re}(\ell_i(\vec{z})), & \text{if } h = H_i. \end{cases}$$

Note that $(\alpha(H_i), \alpha(H_i^{\Re})) \in \{(0, 0), (+, 0), (-, 0), (1, +), (1, -)\}$ for all $1 \leq i \leq n$.

Example 3.4.5. There is a unique complexified hyperplane arrangement in \mathbb{C} , namely $\mathcal{A} = \{H_0 = \{\vec{0}\}\}$. In this case $\{\alpha_{\vec{z}} : \vec{z} \in \mathbb{C}\} = \{(0, 0), (+, 0), (-, 0), (1, +), (1, -)\}$ is the set of covectors of the interval greedoid (E, \mathcal{F}) with $E = \{H_0, H_0^{\Re}\}$ and $\mathcal{F} = \{\emptyset, \{H_0^{\Re}\}, \{H_0, H_0^{\Re}\}\}$ (cf. Example 3.2.4). Figure 7 illustrates the partial order on these covectors. \circ

Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a complexified hyperplane arrangement in \mathbb{C}^d . Let \mathcal{L} be the lattice of all intersections of subspaces from the set

$$E_{\mathcal{A}} = \{H_1, \dots, H_n, H_1^{\Re}, \dots, H_n^{\Re}\},$$

ordered by inclusion. Then \mathcal{L} is a lower semimodular lattice and $E_{\mathcal{A}}$ is the set of meet-irreducible elements of \mathcal{L} [BZ92a]. By Proposition 2.4.1, $(E_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ is an interval greedoid, where

$$\mathcal{F}_{\mathcal{A}} = \{\{h_1, h_2, \dots, h_k\} \subseteq E_{\mathcal{A}} : \mathbb{C}^d \succ h_1 \succ \dots \succ (h_1 \cap h_2 \cap \dots \cap h_k)\}.$$

Lemma 3.4.6. *Let \mathcal{A} be a complexified hyperplane arrangement, and let $(E_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ be the interval greedoid as defined above. Then, for $X \in \mathcal{F}_{\mathcal{A}}$, we have*

$$(3.1) \quad \xi(X) = \left\{ h \in E : \bigcap_{h' \in X} h' \subseteq h \right\},$$

(3.2)

$$\Gamma(X) = \left\{ H_e^{\Re} \in E : \bigcap_{h \in X} h \not\subseteq H_e^{\Re} \right\} \cup \left\{ H_e \in E : \bigcap_{h \in X} h \subseteq H_e^{\Re} \text{ and } \bigcap_{h \in X} h \not\subseteq H_e \right\}.$$

Proof. (3.1) follows directly from Example 2.6.16. We now show (3.2). Let $M = \bigcap_{h \in X} h \subset \mathbb{C}^d$. Thinking of \mathbb{C}^d as a $2d$ -dimensional real vector space, we can decompose it into real and complex parts as $\mathbb{C}^d = \Re(\mathbb{C}^d) \oplus \Im(\mathbb{C}^d)$, where each of the summands is a d -dimensional real vector space, and multiplication by i provides an isomorphism from $\Re(\mathbb{C}^d)$ to $\Im(\mathbb{C}^d)$. Note that H_i and H_i^{\Re} can also be expressed as a direct sum of a real and a complex part. (This relies on the fact that our arrangement is a complexified real arrangement, rather than being an arbitrary

complex arrangement.) Note further that in either case, the imaginary part corresponds to a subspace of the real part. It follows that M , also, can be written as $M = \Re(M) \oplus \Im(M)$, with $\Im(M)$ a subspace of $\Re(M)$.

Observe first that if $H_e^{\Re} \not\geq M$, then, since H_e^{\Re} is real codimension one in \mathbb{C}^d , we have $M \succ M \cap H_e^{\Re}$, so $H_e^{\Re} \in \Gamma(X)$. Also, in this case, we have $M \cap H_e^{\Re} \succ M \cap H_e$, because $\Theta_e \not\geq \Im(M)$, and thus $\Theta_e \not\geq \Re(M)$ either. It follows that in this case $H_e^{\Re} \in \Gamma(X)$ and $H_e \notin \Gamma(X)$.

Finally, if $H_e^{\Re} \geq M$, we observe that H_e is codimension one in H_e^{\Re} , and thus that either $H_e \geq M$ or $M \succ M \cap H_e$. This completes the proof of the lemma. \square

Lemma 3.4.7. *Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a complexified hyperplane arrangement in \mathbb{C}^d . Then $\alpha_{\bar{z}}$ is a covector over the interval greedoid $(E_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$, for every $\bar{z} \in \mathbb{C}^d$.*

Proof. Recall that a map $\alpha : E \rightarrow \{0, +, -, 1\}$ is a covector of an interval greedoid (E, \mathcal{F}) if and only if there exists $X \in \mathcal{F}$ such that $\alpha(e) = 0$ if and only if $e \in \xi(X)$, $\alpha(e) = \pm$ if and only if $e \in \Gamma(X)$, and $\alpha(e) = 1$ otherwise.

Let $\alpha = \alpha_{\bar{z}}$ be defined as above. Observe that $\alpha(h) = 0$ if and only if $\bar{z} \in h$. Let

$$A = \{h \in E : \alpha(h) = 0\} = \{H_i : \bar{z} \in H_i\} \cup \{H_i^{\Re} : \bar{z} \in H_i^{\Re}\} \subseteq E$$

and let X be maximal among the elements of $\mathcal{F}_{\mathcal{A}}$ contained in A .

We show that $\alpha(h) = 0$ if and only if $h \in \xi(X)$ by showing that $\xi(X) = A$. Suppose $h \in \xi(X)$. Then $h \supseteq \cap_{h' \in X} h'$. Since $X \subseteq A$, it follows that $\alpha(h') = 0$ for all $h' \in X$. Thus, $\bar{z} \in h'$ for all $h' \in X$. It follows that $\bar{z} \in h$. Thus, $h \in A$.

Conversely, suppose $h \in A$. Then $\alpha(h) = 0$. If $h = H_i^{\Re}$, then $\{h\} \in \mathcal{F}_{\mathcal{A}}$, so we can augment $\{h\}$ from X until we get a set Y of cardinality $|X|$. Since X is maximal among the feasible sets contained in A and $|X| = |Y|$, Y is maximal as well. Thus, $X \sim Y$, so $Y \subseteq \xi(X)$. In particular, $h \in \xi(X)$. On the other hand, if $h = H_i$, then $\ell_i(\bar{z}) = 0$, so $H_i^{\Re} \in A$ also. Since $\{H_i, H_i^{\Re}\} \in \mathcal{F}_{\mathcal{A}}$, the same argument shows that $h \in \xi(X)$. Thus, $A \subseteq \xi(X)$.

Next we show that $\alpha(h) \in \{+, -\}$ if and only if $\alpha(h) \in \Gamma(X)$. Let $h \in \Gamma(X)$. Since $\xi(X)$ and $\Gamma(X)$ are disjoint, it follows from the above that $\alpha(h) \neq 0$. So it suffices to show that $\alpha(h) \in \{0, +, -\}$. By construction, this is true for $h = H_i^{\Re}$ since $\alpha(H_i^{\Re}) = \sigma_{\Im}(\ell_i(\bar{z})) \in \{0, +, -\}$. If $h = H_i$, then, by the above description of $\Gamma(X)$, we have $H_i^{\Re} \in \xi(X)$. So, $\sigma_{\Im}(\ell_i(\bar{z})) = 0$, which implies $\alpha(h) = \sigma_{\Re}(\ell_i(\bar{z})) \in \{0, +, -\}$.

Conversely, suppose $\alpha(h) \in \{+, -\}$. Since $\alpha(h) \neq 0$, we have $h \notin \xi(X)$, or equivalently, $\bigcap_{x \in X} x \not\supseteq h$. So if $h = H_i^{\Re}$, then $h \in \Gamma(X)$. If $h = H_i$, then we need to show that $H_i^{\Re} \supseteq \bigcap_{x \in X} x$, or equivalently, $H_i^{\Re} \in \xi(X)$. Well, $\sigma_{\Re}(\ell_i(\bar{z})) = \alpha(H_i) \in \{+, -\}$, so $\sigma_{\Im}(\ell_i(\bar{z})) = 0$. This implies $H_i^{\Re} \in \xi(X)$. Hence, $h = H_i \in \Gamma(X)$.

Finally, it follows from the above that $\alpha(h) = 1$ if and only if $h \notin \Gamma(X) \cup \xi(X)$. Therefore, α is a covector of $(E_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$. \square

Remark 3.4.8. As in Example 3.2.4, the product of two covectors α and β can be computed component-wise, or pair-wise using the identity:

$$\left((\alpha \circ \beta)(H_i), (\alpha \circ \beta)(H_i^{\Re}) \right)$$

$$= \begin{cases} (\beta(H_i), \beta(H_i^{\Re})), & \text{if } (\beta(H_i), \beta(H_i^{\Re})) > (\alpha(H_i), \alpha(H_i^{\Re})), \\ (\alpha(H_i), \alpha(H_i^{\Re})), & \text{otherwise,} \end{cases}$$

where the comparison $(\beta(H_i), \beta(H_i^{\Re})) > (\alpha(H_i), \alpha(H_i^{\Re}))$ is performed in the poset illustrated in Figure 7.

Theorem 3.4.9. *If $\mathcal{A} = \{H_1, \dots, H_n\}$ is a complexified hyperplane arrangement in \mathbb{C}^d , then $\mathcal{G} = \{\alpha_{\vec{z}} : \vec{z} \in \mathbb{C}^d\}$ is an oriented interval greedoid over $(E_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$.*

Proof. We show that \mathcal{G} satisfies (OG1)–(OG4).

(OG1) Suppose $[X]$ is a flat of $(E_{\mathcal{A}}, \mathcal{F}_{\mathcal{A}})$ for some $X \in \mathcal{F}_{\mathcal{A}}$. Let \vec{z} be a generic point of $\bigcap_{x \in X} x$. The support of $\alpha_{\vec{z}}$ is the flat $[Y]$ such that Y is maximal among feasible sets contained in $\{h : \alpha_{\vec{z}}(h) = 0\}$. Since \vec{z} is generic, this set is equal to $\xi(X)$. It follows that Y is equivalent to X , so $[X] = [Y] = \text{supp}(\alpha_{\vec{z}})$.

(OG2) Suppose $\alpha_{\vec{z}} \in \mathcal{G}$. Then $-\alpha_{\vec{z}} = \alpha_{-\vec{z}}$ because $\sigma_i(\ell_j(-\vec{z})) = \sigma_i(-\ell_j(\vec{z}))$. So $-\alpha_{\vec{z}} \in \mathcal{G}$.

(OG3) Let $\alpha_{\vec{x}}, \alpha_{\vec{y}} \in \mathcal{G}$. For sufficiently small $t > 0$, we have $\sigma_{\Re}(\vec{u} + t\vec{v}) = \sigma_{\Re}(\vec{u}) \star \sigma_{\Re}(\vec{v})$ and $\sigma_{\Im}(\vec{u} + t\vec{v}) = \sigma_{\Im}(\vec{u}) \star \sigma_{\Im}(\vec{v})$. It follows that $\alpha_{\vec{x}+t\vec{y}} = \alpha_{\vec{x}} \circ \alpha_{\vec{y}}$ for a sufficiently small $t > 0$.

(OG4) Let $\alpha, \beta \in \mathcal{G}$ and $h \in S(\alpha, \beta)$ such that $(\alpha \circ \beta)(h) \neq 1$. Pick $\vec{x}, \vec{y} \in \mathbb{C}^d$ such that $\alpha = \alpha_{\vec{x}}$ and $\beta = \alpha_{\vec{y}}$. We can assume for all $1 \leq i \leq n$ that the line $t\vec{x} + (1-t)\vec{y}$, for $0 < t < 1$, does not intersect H_i if \vec{x} and \vec{y} are not both contained in H_i^{\Re} (otherwise perturb \vec{x} and \vec{y} slightly).

Since $h \in S(\alpha_{\vec{x}}, \alpha_{\vec{y}})$, we have $\alpha_{\vec{x}}(h) = -\alpha_{\vec{y}}(h) \in \{+, -\}$. Hence, $\Re(\ell_h(\vec{x}))$ and $\Re(\ell_h(\vec{y}))$ or $\Im(\ell_h(\vec{x}))$ and $\Im(\ell_h(\vec{y}))$ have opposite signs, where ℓ_h is the form associated to h (that is, $h = \ker(\ell_h)$ or $h = \ker(\ell_h)^{\Re}$). So there exists $0 < t < 1$ such that the real part (or imaginary part) of $\ell_h(t\vec{x} + (1-t)\vec{y})$ is zero. Let $\gamma = \alpha_{t\vec{x}+(1-t)\vec{y}}$. Then $\gamma(h) = 0$.

Let $e \notin S(\alpha_{\vec{x}}, \alpha_{\vec{y}})$ and $(\alpha_{\vec{x}} \circ \alpha_{\vec{y}})(e) \neq 1$. Suppose first that $e = H_i$ for some i . Then $\Im(\ell_i(\vec{x})) = 0 = \Im(\ell_i(\vec{y}))$, for otherwise $(\alpha_{\vec{x}} \circ \alpha_{\vec{y}})(e) = 1$. This implies that $\Im(t\ell_i(\vec{x}) + (1-t)\ell_i(\vec{y})) = 0$, so $\gamma(e) = \sigma_{\Re}(\ell_i(t\vec{x} + (1-t)\vec{y}))$ is the sign of

$$\Re(\ell_i(t\vec{x} + (1-t)\vec{y})) = t\Re(\ell_i(\vec{x})) + (1-t)\Re(\ell_i(\vec{y})).$$

Since both of the coefficients t and $(1-t)$ are positive and since $\Re(\ell_i(\vec{x}))$ and $\Re(\ell_i(\vec{y}))$ are not of opposite signs, it follows that $\gamma(e)$ is the sign of $\Re(\ell_i(\vec{x}))$ if it is nonzero and the sign of $\Re(\ell_i(\vec{y}))$ otherwise. This is precisely $(\alpha_{\vec{x}} \circ \alpha_{\vec{y}})(e)$. Similarly, if $e = H_i^{\Re} \notin S(\alpha_{\vec{x}}, \alpha_{\vec{y}})$, then $\gamma(e) = (\alpha_{\vec{x}} \circ \alpha_{\vec{y}})(e)$. \square

4. RESTRICTION AND CONTRACTION OF ORIENTED INTERVAL GREEDOIDS

4.1. Contraction. This section introduces an operation on oriented interval greedoids that produces an oriented interval greedoid on the contraction of the underlying interval greedoid. We begin by studying the relationship between an interval greedoid and its contractions.

4.1.1. *Contraction of interval greedoids.* Let (E, \mathcal{F}) denote an interval greedoid and Φ its lattice of flats. Recall that for $X \in \mathcal{F}$, the **contraction** of (E, \mathcal{F}) by X is the interval greedoid with feasible sets

$$\mathcal{F}/X = \{Y \subseteq E \setminus X : Y \cup X \in \mathcal{F}\}$$

and ground set $\bigcup_{Y \in \mathcal{F}/X} Y$. We let Φ/X , Γ/X and ξ/X denote the corresponding notions in the contraction. For $Y \in \mathcal{F}/X$, we let $(\Phi/X)(Y)$ denote the flat in the contraction that contains Y .

Proposition 4.1.1. *Suppose (E, \mathcal{F}) is an interval greedoid and $X \in \mathcal{F}$. Then*

- (1) $\Phi/X \cong [\hat{0}, [X]] \subseteq \Phi$.
- (2) If $Y \in \mathcal{F}/X$, then $(\Gamma/X)(Y) = \Gamma(X \cup Y)$.
- (3) If $Y \in \mathcal{F}/X$, then $(\xi/X)(Y) \subseteq \xi(Y \cup X) \cap \bigcup_{Z \in \mathcal{F}/X} Z$.

Proof. (1) Define a map $\Phi/X \rightarrow [\hat{0}, [X]]$ by mapping the flat containing Y (in the contraction \mathcal{F}/X) to the flat $[Y \cup X]$ of (E, \mathcal{F}) . The fact that this map is well-defined follows from the identity: $(\mathcal{F}/X)/Y = \mathcal{F}/(X \cup Y)$ for $X \in \mathcal{F}$ and $Y \in \mathcal{F}/X$. This identity also implies that the map is injective. It remains to show that the map is surjective. Let $[Z] \leq [X]$. Then $\xi(X) \subseteq \xi(Z)$. Hence, there exists Z' containing X with Z' maximal among the feasible sets contained in $\xi(Z)$. Therefore, $Z' \setminus X \in \mathcal{F}/X$, and $Z' \setminus X$ maps to the flat containing $(Z' \setminus X) \cup X = Z'$, which is $[Z]$ by Proposition 2.6.1.

(2) Suppose $x \in (\Gamma/X)(Y)$. Then $Y \cup x \in \mathcal{F}/X$. So $(X \cup Y) \cup x \in \mathcal{F}$. That is, $x \in \Gamma(X \cup Y)$. Conversely, suppose $x \in \Gamma(X \cup Y)$. Then $X \cup (Y \cup x) \in \mathcal{F}$, and so $(Y \cup x) \in \mathcal{F}/X$. That is, $x \in (\Gamma/X)(Y)$.

(3) Let $x \in (\xi/X)(Y)$. Then $x \in W$ for some $W \in \mathcal{F}/X$ that is equivalent (in \mathcal{F}/X) to Y . So $x \in \bigcup_{Z \in \mathcal{F}/X} Z$. And since the map defined in (1) is well-defined, we have $[X \cup Y] = [X \cup W]$. Hence, $x \in W \subseteq \xi(X \cup W) = \xi(X \cup Y)$. \square

We remark that the containment in the previous result can be proper.

4.1.2. *Contractions of oriented interval greedoids.* Let $(E, \mathcal{F}, \mathcal{G})$ be an oriented interval greedoid and $\Phi = \text{supp}(\mathcal{G})$ the lattice of flats of (E, \mathcal{F}) . For $A \in \Phi$, let

$$\mathcal{G}_{\leq A} = \{\alpha \in \mathcal{G} : \text{supp}(\alpha) \leq A\}.$$

Then $\mathcal{G}_{\leq A}$ is a subsemigroup of \mathcal{G} . We'll show that it is isomorphic to an oriented interval greedoid over the contraction of (E, \mathcal{F}) by $X \in \mathcal{F}$, where $A = [X]$.

Let α be a covector of (E, \mathcal{F}) with $\text{supp}(\alpha) \leq [X]$. By definition of the partial order, there exists $Y \in \mathcal{F}/X$ such that $\text{supp}(\alpha) = [X \cup Y]$. Therefore, Y is a feasible set in the contracted interval greedoid and so it makes sense to talk about its flat $(\Phi/X)(Y)$. By restricting α to the subset $(\Gamma/X)(Y)$, we get a signed flat $((\Phi/X)(Y), \alpha|_{(\Gamma/X)(Y)})$ of the contracted interval greedoid. We denote the covector of this signed flat by $\text{con}_X(\alpha)$. Then,

$$(4.1) \quad \text{con}_X(\alpha)(e) = \begin{cases} 0, & \text{if } e \in (\xi/X)(Y), \\ \alpha(e), & \text{if } e \in (\Gamma/X)(Y), \\ 1, & \text{otherwise.} \end{cases}$$

It follows from Proposition 4.1.1 that if $\text{con}_X(\alpha)(e) \neq 1$, then $\text{con}_X(\alpha)(e) = \alpha(e)$.

Lemma 4.1.2. *Suppose (E, \mathcal{F}) is an interval greedoid and let $X \in \mathcal{F}$. Let α and β be covectors of (E, \mathcal{F}) with $\text{supp}(\alpha), \text{supp}(\beta) \leq [X]$.*

- (1) $(\text{supp}/X)(\text{con}_X(\alpha)) = (\Phi/X)(Y)$ and $(\text{supp}/X)(\text{con}_X(\beta)) = (\Phi/X)(Z)$,
where $Y, Z \in \mathcal{F}/X$ satisfy $[X \cup Y] = \text{supp}(\alpha)$ and $[X \cup Z] = \text{supp}(\beta)$.
- (2) $\text{con}_X(\alpha) \circ \text{con}_X(\beta) = \text{con}_X(\alpha \circ \beta)$.

Proof. (1) Since $\text{con}_X(\alpha)$ is the covector of the signed flat $((\Phi/X)(Y), \alpha|_{(\Gamma/X)(Y)})$, it follows from the definition of supp/X that $(\text{supp}/X)(\text{con}_X(\alpha)) = (\Phi/X)(Y)$.

(2) We first argue that the supports of the two elements are the same. It follows from the definition of \circ that the support of $\text{con}_X(\alpha) \circ \text{con}_X(\beta)$ is the join of their supports, so it is $(\Phi/X)(Y) \vee (\Phi/X)(Z)$ by (1). Under the isomorphism $\Phi/X \cong [\emptyset, [X]]$, this corresponds to $[X \cup Y] \vee [X \cup Z]$, which we can express as $[X \cup W]$ for some $W \in \mathcal{F}/X$. Hence, $(\Phi/X)(Y) \vee (\Phi/X)(Z) = (\Phi/X)(W)$. Note that $[X \cup W]$ is also the support of $\alpha \circ \beta$, so (1) implies that $(\text{supp}/X)(\text{con}_X(\alpha \circ \beta)) = (\Phi/X)(W)$.

Since both $\text{con}_X(\alpha) \circ \text{con}_X(\beta)$ and $\text{con}_X(\alpha \circ \beta)$ are covectors of support $(\Phi/X)(W)$, to show that they are equal it suffices to show that they agree on $(\Gamma/X)(W)$. Let $e \in (\Gamma/X)(W)$. Then,

$$(\text{con}_X(\alpha) \circ \text{con}_X(\beta))(e) = \begin{cases} \text{con}_X(\beta)(e), & \text{if } \text{con}_X(\beta)(e) > \text{con}_X(\alpha)(e), \\ \text{con}_X(\alpha)(e), & \text{otherwise.} \end{cases}$$

Since $(\text{con}_X(\alpha) \circ \text{con}_X(\beta))(e) \neq 1$, it follows that neither $\text{con}_X(\alpha)(e)$ nor $\text{con}_X(\beta)(e)$ is 1. Hence, $\text{con}_X(\alpha)(e) = \alpha(e)$ and $\text{con}_X(\beta)(e) = \beta(e)$ (see the sentence following (4.1)). Therefore,

$$(\text{con}_X(\alpha) \circ \text{con}_X(\beta))(e) = \begin{cases} \beta(e), & \text{if } \beta(e) > \alpha(e), \\ \alpha(e), & \text{otherwise.} \end{cases}$$

This is precisely $(\alpha \circ \beta)(e)$, which is $\text{con}_X(\alpha \circ \beta)(e)$ by (4.1). \square

Proposition 4.1.3. *Let $(E, \mathcal{F}, \mathcal{G})$ denote an oriented interval greedoid and let $X \in \mathcal{F}$. Then*

$$\mathcal{G}/X = \{\text{con}_X(\alpha) : \alpha \in \mathcal{G} \text{ and } \text{supp}(\alpha) \leq [X]\}$$

defines an oriented interval greedoid over the contraction of (E, \mathcal{F}) by X .

Proof. (OG1). Let $A \in \Phi/X$. Then $A = (\Phi/X)(Y)$ for some $Y \in \mathcal{F}/X$, and so $[Y \cup X] \in \Phi$. Since \mathcal{G} satisfies (OG1), there exists $\alpha \in \mathcal{G}$ with $\text{supp}(\alpha) = [Y \cup X] \leq [X]$. Then $\text{con}_X(\alpha) \in \mathcal{G}/X$ and $(\text{supp}/X)(\text{con}_X(\alpha)) = (\Phi/X)(Y) = A$ by Lemma 4.1.2.

(OG2) Suppose $\nu \in \mathcal{G}/X$. Then there exists some $\beta \in \mathcal{G}$ such that $\text{supp}(\beta) \leq [X]$ and $\text{con}_X(\beta) = \nu$. Then $-\beta \in \mathcal{G}$ by (OG2), and so $-\nu = \text{con}_X(-\beta) \in \mathcal{G}/X$.

(OG3) Suppose $\text{con}_X(\alpha)$ and $\text{con}_X(\beta)$ are in \mathcal{G}/X . Then $\alpha \circ \beta \in \mathcal{G}$, by (OG3), and $\text{supp}(\alpha \circ \beta) = \text{supp}(\alpha) \vee \text{supp}(\beta) \leq [X]$. Therefore, $\text{con}_X(\alpha \circ \beta) \in \mathcal{G}/X$. By Lemma 4.1.2, $\text{con}_X(\alpha \circ \beta) = \text{con}_X(\alpha) \circ \text{con}_X(\beta)$, so $\text{con}_X(\alpha) \circ \text{con}_X(\beta) \in \mathcal{G}/X$.

(OG4) Suppose $\text{con}_X(\alpha), \text{con}_X(\beta) \in \mathcal{G}/X$, and let $x \in (S/X)(\text{con}_X(\alpha), \text{con}_X(\beta))$ such that $(\text{con}_X(\alpha) \circ \text{con}_X(\beta))(x) \neq 1$.

Since $\text{con}_X(\alpha)(x) = -\text{con}_X(\beta)(x) \in \{+, -\}$, it follows from (4.1) that $\alpha(x) = -\beta(x) \in \{+, -\}$. Hence, $x \in S(\alpha, \beta)$. Since $\text{con}_X(\alpha \circ \beta) = \text{con}_X(\alpha) \circ \text{con}_X(\beta)$, it follows that $\text{con}_X(\alpha \circ \beta)(x) \neq 1$, which implies that $(\alpha \circ \beta)(x) \neq 1$. Therefore, (OG4) applies to α, β and x to guarantee the existence of $\gamma \in \mathcal{G}$ satisfying $\gamma(x) = 0$ and for all $y \notin S(\alpha, \beta)$, if $(\alpha \circ \beta)(y) \neq 1$, then $\gamma(y) = (\alpha \circ \beta)(y) = (\beta \circ \alpha)(y)$. We claim that $\text{con}_X(\gamma)$ satisfies the conditions of (OG4) for \mathcal{G}/X .

We first show that $\text{supp}(\gamma) \leq \text{supp}(\alpha \circ \beta)$. Indeed, if $(\alpha \circ \beta)(y) = 0$, then $\alpha(y) = 0$ and $\beta(y) = 0$, so $y \notin S(\alpha, \beta)$. We conclude from (OG4) that $\gamma(y) = (\alpha \circ \beta)(y) = 0$.

Next we argue that $\text{con}_X(\gamma)(x) = 0$. Since $\text{supp}(\gamma) \leq \text{supp}(\alpha \circ \beta)$, it follows that $(\text{supp}/X)(\text{con}_X(\gamma)) \leq (\text{supp}/X)(\text{con}_X(\alpha \circ \beta))$. Then Proposition 3.1.3 and the assumption that $\text{con}_X(\alpha \circ \beta)(x) \neq 1$ implies that $\text{con}_X(\gamma)(x) \neq 1$. If $\text{con}_X(\gamma)(x) \in \{+, -\}$, then $\gamma(x) \in \{+, -\}$ contradicting the fact that $\gamma(x) = 0$. Therefore, $\text{con}_X(\gamma)(x) = 0$.

Now let $y \in \bigcup_{Z \in \mathcal{F}/X} Z$ with $y \notin (S/X)(\text{con}_X(\alpha), \text{con}_X(\beta))$. We claim that $y \notin S(\alpha, \beta)$. If $y \in S(\alpha, \beta)$, then $\alpha(y) = -\beta(y) \in \{+, -\}$, and so $\text{con}_X(\alpha)(y) = \alpha(y) = -\beta(y) = -\text{con}_X(\beta)(y) \in \{+, -\}$ by (4.1), a contradiction.

Now suppose that $\text{con}_X(\alpha \circ \beta)(y) \neq 1$. As above, Proposition 3.1.3 implies that $\text{con}_X(\gamma)(y) \neq 1$. Then the sentence following (4.1) implies that $\text{con}_X(\alpha \circ \beta)(y) = (\alpha \circ \beta)(y)$ and that $\text{con}_X(\gamma)(y) = \gamma(y)$. Hence, $(\alpha \circ \beta)(y) \neq 1$, so $\gamma(y) = (\alpha \circ \beta)(y) = (\beta \circ \alpha)(y)$ by (OG4). Therefore, $\text{con}_X(\gamma)(y) = (\text{con}_X(\alpha) \circ \text{con}_X(\beta))(y) = (\text{con}_X(\beta) \circ \text{con}_X(\alpha))(y)$. \square

The following result identifies \mathcal{G}/X with a subsemigroup of \mathcal{G} .

Proposition 4.1.4. *Let $(E, \mathcal{F}, \mathcal{G})$ denote an oriented interval greedoid and let $X \in \mathcal{F}$. Then there is a semigroup isomorphism*

$$\mathcal{G}_{\leq [X]} \cong \mathcal{G}/X$$

given by mapping $\alpha \in \mathcal{G}$ with $\text{supp}(\alpha) \leq [X]$ to $\text{con}_X(\alpha)$.

Proof. Lemma 4.1.2 shows this is a semigroup morphism. The morphism is surjective by definition of \mathcal{G}/X . It remains to show that the morphism is injective. Suppose $\text{con}_X(\alpha) = \text{con}_X(\beta)$. Then $\text{supp}(\alpha) = \text{supp}(\beta)$, which can be written as $[X \cup Y]$. Now $(\Gamma/X)(Y) = \Gamma(X \cup Y)$, so α and β agree on $\Gamma(X \cup Y)$, and they each are zero on exactly $\xi([X \cup Y])$, so α and β agree, as desired. \square

4.2. Restriction. We introduce a restriction operation for an oriented interval greedoid $(E, \mathcal{F}, \mathcal{G})$ that produces an oriented interval greedoid on a restriction of the interval greedoid (E, \mathcal{F}) . We begin by recalling restriction for interval greedoids.

4.2.1. Restriction of an interval greedoid. Let (E, \mathcal{F}) denote an interval greedoid and Φ its lattice of flats. If $W \subseteq E$ is an arbitrary subset, then the **restriction** of (E, \mathcal{F}) to W is the interval greedoid $(W, \mathcal{F}|_W)$, where

$$\mathcal{F}|_W = \{X \in \mathcal{F} : X \subseteq W\}.$$

To distinguish between objects defined for (E, \mathcal{F}) and $(W, \mathcal{F}|_W)$, we take the following convention. If Ξ is an object defined for (E, \mathcal{F}) (for example, its lattice of

flats Φ , the set of continuations Γ), then $\Xi|_W$ will denote the corresponding object defined for $(W, \mathcal{F}|_W)$ (for example, $\Phi|_W, \Gamma|_W$).

There is a map $\Phi \rightarrow \Phi|_W$ that maps a flat $C \in \Phi$ onto the flat $\mu|_W(W \cap \xi(C))$. We denote the image of C by $C|_W$. Note that if $Y \in \mathcal{F}|_W$, then $Y \in \mathcal{F}$ and the image of $[Y] \in \Phi$ under $\Phi \rightarrow \Phi|_W$ is the flat in $\Phi|_W$ that contains Y , which by our above convention is denoted by $[Y]|_W$.

Lemma 4.2.1. *Suppose (E, \mathcal{F}) is an interval greedoid and let $W \subseteq E$.*

- (1) *If $Y \in \mathcal{F}|_W$, then $\Gamma|_W(Y) = W \cap \Gamma(Y)$.*
- (2) *If $Y \in \mathcal{F}|_W$ and $\xi(Y) \subseteq W$, then $\xi|_W(Y) = \xi(Y)$.*
- (3) *If $A \in \Phi$, then $\Gamma|_W(A|_W) \subseteq W \cap \Gamma(A)$.*
- (4) *If $A \in \Phi$, then $\xi|_W(A|_W) \subseteq W \cap \xi(A)$.*

Proof. (1) If $Y \in \mathcal{F}|_W$, then $\Gamma|_W(Y) = \{y \in W \setminus Y : Y \cup y \in \mathcal{F}\} = W \cap \{y \in E \setminus Y : Y \cup y \in \mathcal{F}\} = W \cap \Gamma(Y)$.

(2) Suppose $Y \in \mathcal{F}|_W$ and $\xi(Y) \subseteq W$. The latter assumption implies that all feasible sets that are equivalent to Y in (E, \mathcal{F}) are contained in W . So they are contained in $\mathcal{F}|_W$. Moreover, they are also equivalent in $\mathcal{F}|_W$ since they are all maximal among the feasible sets contained in $\xi(Y)$ (see Proposition 2.6.1).

(3) Let $Z \in A|_W$ and let $x \in \Gamma|_W(A|_W) = \Gamma|_W(Z)$. By (1), $x \in W \cap \Gamma(Z)$. Since $A|_W = \mu|_W(W \cap \xi(A))$, there exists $Y \in \mathcal{F}$ containing Z that is maximal among the feasible sets contained in $\xi(A)$. Thus, $[Z] \geq [Y] = A$, and by Proposition 3.1.3,

$$\Gamma|_W(Z) = (\Gamma(Z) \cap W) \subseteq (\Gamma(A) \cap W) \cup (\xi(A) \cap W).$$

If $x \in W \cap \xi(A)$, then $Z \cup x \in W \cap \xi(A)$, contradicting that Z is maximal among the feasible sets contained in $W \cap \xi(A)$. Therefore, $x \in W \cap \Gamma(A)$.

(4) By definition $A|_W = \mu|_W(W \cap \xi(A))$, so the sets contained in $A|_W$ are the sets that are maximal among the feasible sets contained in $W \cap \xi(A)$. Let D be a maximal feasible set in $W \cap \xi(A)$, and let C be a set in W that is equivalent to D in the restriction. We want to show that C is contained in $\xi(A)$.

Since C and D are equivalent in the restriction, they have the same continuations inside W . Let x be a continuation of C with $x \notin W$. Then D can be augmented from $C \cup x$, and clearly D can't be augmented from C , so it can be augmented by x . Thus $\Gamma(C)$ contains $\Gamma(D)$, and the converse is also true. So C and D have the same continuations in the original interval greedoid, and therefore are equivalent. In particular, C is in $\xi(A)$ as well. \square

Remark 4.2.2. The inclusions in (3) and (4) can be proper, as can be seen in the following example. Let $E = \{a, b, c\}$ and $\mathcal{F} = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}\}$. Then (E, \mathcal{F}) is an interval greedoid. If $W = \{b, c\}$, then $\mathcal{F}|_W = \{\emptyset\}$, so

$$\begin{aligned} \Gamma|_W([\{a\}]|_W) &= \Gamma|_W(\emptyset) = \emptyset \subsetneq \Gamma([\{a\}]) \cap W = \{b, c\}, \\ \xi|_W([\{a, b\}]|_W) &= \xi|_W(\emptyset) = \emptyset \subsetneq \xi([\{a, b\}]) \cap W = \{b, c\}. \end{aligned}$$

Proposition 4.2.3. *The map $\Phi \rightarrow \Phi|_W$ defined by $C \mapsto C|_W = \mu|_W(W \cap \xi(C))$ for all $C \in \Phi$ is order-preserving, surjective and preserves joins: $(A \vee B)|_W = A|_W \vee B|_W$ for all $A, B \in \Phi$.*

Proof. The mapping is order-preserving since ξ and $\mu|_W$ are order-reversing. The map is surjective since if $[Y]|_W \in \Phi|_W$, then it follows that $Y \in \mathcal{F}$ and that Y is maximal among the feasible subsets contained in $W \cap \xi([Y])$. So, $[Y] \mapsto [Y]|_W$ under this mapping.

Since $A, B \leq A \vee B$, and since the map is order-preserving, $A|_W \vee B|_W \leq (A \vee B)|_W$. Since $A|_W \vee B|_W \in \Phi|_W$, there exists $Z \in \mathcal{F}|_W$ such that $A|_W \vee B|_W = [Z]|_W$. Then Z is maximal among the feasible sets contained in $\xi|_W(A|_W) \cap \xi|_W(B|_W)$ by definition of \vee (Proposition 2.6.17). Since $A|_W$ is the collection of sets that are maximal among the feasible sets contained in $W \cap \xi(A)$, it follows that $\xi|_W(A|_W) \subseteq W \cap \xi(A)$. Therefore, $\xi|_W(A|_W) \cap \xi|_W(B|_W) \subseteq W \cap \xi(A) \cap \xi(B)$. So there exists Y containing Z that is maximal among the feasible sets contained in $W \cap \xi(A) \cap \xi(B)$. We now argue that Y is maximal among the feasible sets contained in $W \cap \xi(A \vee B)$. There exists U containing Y that is maximal among the feasible sets contained in $\xi(A) \cap \xi(B)$. Thus, $[Y] \geq [U] = A \vee B$. This implies $Y \subseteq \xi(Y) \subseteq \xi(A \vee B)$ since ξ is order-reversing. So $Y \subseteq W \cap \xi(A \vee B)$. Since $\xi(A \vee B) \subseteq \xi(A) \cap \xi(B)$, it follows that Y is maximal among the feasible sets contained in $W \cap \xi(A \vee B)$. Therefore, $(A \vee B)|_W = [Y]|_W$. Since $Y \supseteq Z$, we have $[Y]|_W \leq [Z]|_W$. Thus, $(A \vee B)|_W \leq A|_W \vee B|_W$. Therefore, $(A \vee B)|_W = A|_W \vee B|_W$. \square

Since $\mathcal{F}|_W \subseteq \mathcal{F}$, there is also a map in the reverse direction $\Phi|_W \rightarrow \Phi$, defined by $A \mapsto [Y]$ for any $Y \in A$. Proposition 2.6.1 implies the map is well-defined, and the identity $(\mathcal{F}|_W)/Y = (\mathcal{F}/Y)|_W = \{X \subseteq W \setminus Y : X \cup Y \in \mathcal{F}\}$ implies the map is injective. It is order-preserving and its image is contained in the interval $[[X], \hat{1}]$, where X is maximal among the feasible sets contained in W .

Unlike for matroids, for an arbitrary interval greedoid, the lattice of flats $\Phi|_W$ is not, in general, an interval of Φ . However, if $W \supseteq \xi(X)$, where X is maximal among the feasible subsets contained in W , then $\Phi|_W \cong [[X], \hat{1}] \subseteq \Phi$. (This is obtained by considering the compositions of the maps defined above.)

4.2.2. Restricting Covectors. Let $W \subseteq E$ and let α be a covector of (E, \mathcal{F}) . Let $A = \text{supp}(\alpha)$. It follows from Lemma 4.2.1 that $\Gamma|_W(A|_W) \subseteq \Gamma(A)$, so $(A|_W, \alpha|_{\Gamma|_W(A|_W)})$ is a signed flat of $(W, \mathcal{F}|_W)$. Let $\text{res}_W(\alpha)$ denote the covector of this signed flat:

$$(4.2) \quad \text{res}_W(\alpha)(w) = \begin{cases} 0, & \text{if } w \in \xi|_W(A|_W), \\ \alpha(w), & \text{if } w \in \Gamma|_W(A|_W), \\ 1, & \text{otherwise,} \end{cases}$$

for all $w \in W$. Observe that by construction $\text{supp}|_W(\text{res}_W(\alpha)) = \text{supp}(\alpha)|_W$. Also note that by Lemma 4.2.1, if $\text{res}_W(\alpha)(w) \neq \alpha(w)$, then $\text{res}_W(\alpha)(w) = 1$.

Example 4.2.4 (Antimatroid from three colinear points). Let $(E, \mathcal{F}, \mathcal{G})$ be the oriented interval greedoid arising from the convex geometry of three colinear points, x, y, z with y between x, z . Let $W = \{x, y\}$. The covectors of $\mathcal{G}|_W$ are: $(\pm, 1); (0, \pm); (0, 0)$. For $\alpha \in \mathcal{G}$, if $\alpha(z) \neq 0$, then $\text{res}_W(\alpha)$ equals the restriction of α to W . However, if, for example, $\alpha = (+, +, 0)$, then $\text{res}_W(\alpha) = (+, 1)$. \circ

As we have just seen in an example, $\text{res}_W(\alpha)$ cannot necessarily be obtained by restricting α to W . The following proposition sheds more light on this.

Proposition 4.2.5. *Suppose (E, \mathcal{F}) is an interval greedoid and let $W \subseteq E$.*

- (1) If α is a covector of (E, \mathcal{F}) and $A = \text{supp}(\alpha)$, then $\text{res}_W(\alpha) = \alpha|_W$ if and only if $\xi|_W(A|_W) = W \cap \xi(A)$ and $\Gamma|_W(A|_W) = W \cap \Gamma(A)$.
- (2) If α and β are covectors of (E, \mathcal{F}) , then $\text{res}_W(\alpha \circ \beta) = \text{res}_W(\alpha) \circ \text{res}_W(\beta)$.

Proof. (1) This is obvious from the definitions.

(2) Let $A = \text{supp}(\alpha)$ and $B = \text{supp}(\beta)$. By §4.2.1, $A|_W \vee B|_W = (A \vee B)|_W$. Thus, $\text{res}_W(\alpha) \circ \text{res}_W(\beta)$ and $\text{res}_W(\alpha \circ \beta)$ have the same support. It is therefore clear that they coincide. \square

4.2.3. Restriction of an oriented interval greedoid. The following results shows that the covectors obtained by restricting the covectors of an oriented interval greedoid $(E, \mathcal{F}, \mathcal{G})$ satisfy the first three axioms for an oriented interval greedoid.

Proposition 4.2.6. *Suppose $(E, \mathcal{F}, \mathcal{G})$ is an oriented interval greedoid. If $W \subseteq E$, then $(W, \mathcal{F}|_W, \mathcal{G}|_W)$ satisfies (OG1), (OG2) and (OG3), where*

$$\mathcal{G}|_W = \{\text{res}_W(\alpha) : \alpha \in \mathcal{G}\}.$$

Proof. By construction, we have that $\mathcal{G}|_W$ is a collection of covectors of $(W, \mathcal{F}|_W)$ and that $\text{supp}|_W(\text{res}_W(\alpha)) = \text{supp}(\alpha)|_W \in \Phi|_W$.

(OG1) Let $A \in \Phi|_W$ and let $Y \in A$. Then $[Y] \in \Phi$ and so there exists $\alpha \in \mathcal{G}$ with $\text{supp}(\alpha) = [Y]$. So $\text{res}_W(\alpha) \in \mathcal{G}|_W$ and $\text{supp}|_W(\text{res}_W(\alpha)) = \text{supp}(\alpha)|_W = A|_W$.

(OG2) If $\text{res}_W(\alpha) \in \mathcal{G}|_W$, then $-\alpha \in \mathcal{G}$. Hence, $-\text{res}_W(\alpha) = \text{res}_W(-\alpha) \in \mathcal{G}|_W$.

(OG3) Suppose $\text{res}_W(\alpha), \text{res}_W(\beta) \in \mathcal{G}|_W$ and let $A = \text{supp}(\alpha)$ and $B = \text{supp}(\beta)$. Then $\alpha \circ \beta \in \mathcal{G}$, and so $\text{res}_W(\alpha \circ \beta) \in \mathcal{G}|_W$. By Proposition 4.2.5, $\text{res}_W(\alpha \circ \beta) = \text{res}_W(\alpha) \circ \text{res}_W(\beta)$, so $\text{res}_W(\alpha) \circ \text{res}_W(\beta) \in \mathcal{G}|_W$. \square

Although (OG4) may not hold for an arbitrary restriction, it does hold for certain restrictions, so we get an oriented interval greedoid.

Theorem 4.2.7. *Suppose $(E, \mathcal{F}, \mathcal{G})$ is an oriented interval greedoid and let $W \subseteq E$. If $\text{res}_W(\alpha) = \alpha|_W$ for all $\alpha \in \mathcal{G}$, then $(W, \mathcal{F}|_W, \mathcal{G}|_W)$ is an oriented interval greedoid.*

Proof. (OG1)–(OG3) hold by Proposition 4.2.6. The assumption that $\text{res}_W(\alpha) = \alpha|_W$ for all $\alpha \in \mathcal{G}$ means that (OG4) for \mathcal{G} implies (OG4) for $(W, \mathcal{F}|_W, \mathcal{G}|_W)$. \square

§4.3 and §4.4 describe restriction to two particular types of subsets of E .

4.3. Restriction to $\Gamma(\emptyset)$. In this section we treat restriction to $\Gamma(\emptyset)$.

Proposition 4.3.1. *Suppose $(E, \mathcal{F}, \mathcal{G})$ is an oriented interval greedoid. Then the restriction $(\Gamma(\emptyset), \mathcal{F}|_{\Gamma(\emptyset)}, \mathcal{G}|_{\Gamma(\emptyset)})$ is an oriented matroid, and*

$$\mathcal{G}|_{\Gamma(\emptyset)} = \{\alpha|_{\Gamma(\emptyset)} : \alpha \in \mathcal{G}\}.$$

Proof. Since $(\Gamma(\emptyset), \mathcal{F}|_{\Gamma(\emptyset)})$ is a matroid, it will follow from Theorem 3.4.1 that $(\Gamma(\emptyset), \mathcal{F}|_{\Gamma(\emptyset)}, \mathcal{G}|_{\Gamma(\emptyset)})$ is an oriented matroid once we show that it is an oriented interval greedoid. By Proposition 4.2.5 and Theorem 4.2.7, we need only show that $\Gamma|_{\Gamma(\emptyset)}(A|_{\Gamma(\emptyset)}) = \Gamma(\emptyset) \cap \Gamma(A)$ and $\xi|_{\Gamma(\emptyset)}(A|_{\Gamma(\emptyset)}) = \Gamma(\emptyset) \cap \xi(A)$.

Suppose $x \in \Gamma(\emptyset) \cap \Gamma(A)$. Let $Y \in A|_{\Gamma(\emptyset)}$. Then $Y \subseteq \Gamma(\emptyset) \cap \xi(A)$, so there exists $Z \supseteq Y$ such that Z is maximal among the sets in \mathcal{F} contained in $\xi(A)$. By Proposition 2.6.1, $Z \in A$, so $\Gamma(A) = \Gamma(Z)$. Hence, $Z \cup x \in \mathcal{F}$ since $x \in \Gamma(A)$. Also, $\{x\} \in \mathcal{F}$ since $x \in \Gamma(\emptyset)$. Therefore, (IG3) applied to $\emptyset \subseteq Y \subseteq Z$ implies $Y \cup x \in \mathcal{F}$. Since $Y \cup x \subseteq \Gamma(\emptyset)$, we have $Y \cup x \in \mathcal{F}|_{\Gamma(\emptyset)}$. So $x \in \Gamma|_{\Gamma(\emptyset)}(A|_{\Gamma(\emptyset)})$. This establishes one inclusion. The reverse inclusion follows from Lemma 4.2.1.

It remains to show that $\xi|_{\Gamma(\emptyset)}(A|_{\Gamma(\emptyset)}) = \Gamma(\emptyset) \cap \xi(A)$. Suppose $x \in \Gamma(\emptyset) \cap \xi(A)$. Then $\{x\} \in \mathcal{F}$ since $x \in \Gamma(\emptyset)$. Therefore, there exists Y containing x such that Y is maximal among the feasible sets contained in $\Gamma(\emptyset) \cap \xi(A)$. Then $Y \in A|_{\Gamma(\emptyset)}$. Therefore, $Y \subseteq \xi|_{\Gamma(\emptyset)}(A|_{\Gamma(\emptyset)})$, and so $x \in \xi|_{\Gamma(\emptyset)}(A|_{\Gamma(\emptyset)})$. This, combined with Lemma 4.2.1, establishes the equality $\xi|_{\Gamma(\emptyset)}(A|_{\Gamma(\emptyset)}) = \Gamma(\emptyset) \cap \xi(A)$. \square

4.4. Restriction to $\xi(X)$. To simplify notation, we write $\xi(X)$ for $\xi([X])$ for any feasible set $X \in \mathcal{F}$.

We show that restriction to $\xi(X)$ for $X \in \mathcal{F}$ produces an oriented interval greedoid $(\xi(X), \mathcal{F}|_{\xi(X)}, \mathcal{G}|_{\xi(X)})$ and that there is a semigroup isomorphism

$$\mathcal{G}|_{\xi(X)} \xrightarrow{\cong} \mathcal{G}_{\geq \alpha} = \{\beta \in \mathcal{G} : \beta \geq \alpha\} = \{\alpha \circ \beta : \beta \in \mathcal{G}\},$$

where α is any covector with $\text{supp}(\alpha) = [X]$.

Lemma 4.4.1. *Suppose (E, \mathcal{F}) is an interval greedoid and let $X \in \mathcal{F}$ and $A \in \Phi$.*

- (1) $A|_{\xi(X)} = A \vee [X]$.
- (2) $\xi|_{\xi(X)}(A|_{\xi(X)}) = \xi(A \vee [X]) = \xi(A \vee [X]) \cap \xi(X)$.
- (3) $\Gamma|_{\xi(X)}(A|_{\xi(X)}) = \Gamma(A \vee [X]) \cap \xi(X)$.

Proof. (1) $A|_{\xi(X)}$ is $\mu|_{\xi(X)}(\xi(X) \cap \xi(A))$, the flat that consists of the sets that are maximal among the feasible sets contained in $\xi(X) \cap \xi(A)$. By Proposition 2.6.17, this is $A \vee [X]$.

(2) This follows from (1) since all feasible sets in $A \vee [X]$ are contained in $\xi(X)$.

(3) Write $A \vee [X] = [Y]$ for some $Y \in \mathcal{F}$. Then $[X] \leq [Y]$, so $Y \in \mathcal{F}|_{\xi(X)}$ since $\xi(Y) \subseteq \xi(X)$. Thus, $\Gamma|_{\xi(X)}(A|_{\xi(X)}) = \Gamma|_{\xi(X)}(Y) = \{y \in \xi(X) \setminus Y : Y \cup y \in \mathcal{F}\} = \xi(X) \cap \Gamma(Y) = \xi(X) \cap \Gamma(A \vee [X])$. \square

Lemma 4.4.2. *Suppose $(E, \mathcal{F}, \mathcal{G})$ is an oriented interval greedoid and let $X \in \mathcal{F}$.*

$$\begin{aligned} \mathcal{G}|_{\xi(X)} &= \{\text{res}_{\xi(X)}(\alpha) : \alpha \in \mathcal{G} \text{ and } \text{supp}(\alpha) \geq [X]\} \\ &= \{\alpha|_{\xi(X)} : \alpha \in \mathcal{G} \text{ and } \text{supp}(\alpha) \geq [X]\}. \end{aligned}$$

Proof. We begin by proving the first equality. We show that if $\beta \in \mathcal{G}$, then there exists $\alpha \in \mathcal{G}$ with $\text{supp}(\alpha) \geq [X]$ and $\text{res}_{\xi(X)}(\alpha) = \text{res}_{\xi(X)}(\beta)$. Let $\beta \in \mathcal{G}$ and let $B = \text{supp}(\beta)$. Since \mathcal{G} satisfies (OG1), there exists $\gamma \in \mathcal{G}$ such that $\text{supp}(\gamma) = [X]$. Since $\text{supp}(\gamma \circ \beta) = [X] \vee B$ and $B|_{\xi(X)} = B \vee [X] = (B \vee [X])|_{\xi(X)}$ by Lemma 4.4.1,

$$\text{res}_{\xi(X)}(\gamma \circ \beta) = \begin{cases} 0, & \text{if } x \in \xi|_{\xi(X)}(B|_{\xi(X)}), \\ (\gamma \circ \beta)(x), & \text{if } x \in \Gamma|_{\xi(X)}(B|_{\xi(X)}), \\ 1, & \text{otherwise.} \end{cases}$$

Therefore, $\text{res}_{\xi(X)}(\gamma \circ \beta) = \text{res}_{\xi(X)}(\beta)$ if and only if $(\gamma \circ \beta)(x) = \beta(x)$ for $x \in \Gamma|_{\xi(X)}(B|_{\xi(X)})$. So suppose $x \in \Gamma|_{\xi(X)}(B|_{\xi(X)})$. By Lemma 4.4.1, $x \in \xi(X) \cap \Gamma(B \vee [X])$.

$[X]$), and by Proposition 3.1.3, $x \in \xi(X) \cap (\Gamma(B) \cup \Gamma(X))$. This implies $x \in \Gamma(B)$ because $\xi(X) \cap \Gamma(X) = \emptyset$. Therefore, $(\gamma \circ \beta)(x) = \beta(x)$, and so $\text{res}_{\xi(X)}(\gamma \circ \beta)(x) = \beta(x) = \text{res}_{\xi(X)}(\beta)(x)$.

We now prove the second equality. Let $\text{res}_{\xi(X)}(\alpha)$ such that $\alpha \in \mathcal{G}$ and $A = \text{supp}(\alpha) \geq [X]$. By Lemma 4.4.1 we have $A|_{\xi(X)} = A \vee [X] = A$, $\xi|_{\xi(X)}(A|_{\xi(X)}) = \xi(X) \cap \xi(A)$ and $\Gamma|_{\xi(X)}(A|_{\xi(X)}) = \xi(X) \cap \Gamma(A)$. Therefore, by Proposition 4.2.5, $\text{res}_{\xi(X)}(\alpha) = \alpha|_{\xi(X)}$ for all $\alpha \in \mathcal{G}$ such that $\text{supp}(\alpha) \geq [X]$. \square

Theorem 4.4.3. *Let $(E, \mathcal{F}, \mathcal{G})$ denote an oriented interval greedoid and let $X \in \mathcal{F}$. Then $(\xi(X), \mathcal{F}|_{\xi(X)}, \mathcal{G}|_{\xi(X)})$ is an oriented interval greedoid.*

Proof. By Proposition 4.2.6 we need only show that $\mathcal{G}|_{\xi(X)}$ satisfies (OG4). Let $\text{res}_{\xi(X)}(\alpha)$ and $\text{res}_{\xi(X)}(\beta)$ be covectors in $\mathcal{G}|_{\xi(X)}$. By Lemma 4.4.2, we can assume $\text{supp}(\alpha) \geq [X]$, $\text{supp}(\beta) \geq [X]$, $\text{res}_{\xi(X)}(\alpha) = \alpha|_{\xi(X)}$ and $\text{res}_{\xi(X)}(\beta) = \beta|_{\xi(X)}$.

Let $x \in S(\text{res}_{\xi(X)}(\alpha), \text{res}_{\xi(X)}(\beta))$ with $\text{res}_{\xi(X)}(\alpha \circ \beta)(x) \neq 1$. Then $x \in S(\alpha, \beta)$ and $(\alpha \circ \beta)(x) \neq 1$. By (OG4) applied to \mathcal{G} , there exists $\gamma \in \mathcal{G}$ such that $\gamma(x) = 0$ and for all $y \notin S(\alpha, \beta)$, if $(\alpha \circ \beta)(y) \neq 1$, then $\gamma(y) = (\alpha \circ \beta)(y) = (\beta \circ \alpha)(y)$.

We show that $\text{res}_{\xi(X)}(\gamma)$ satisfies the conditions of (OG4). Let $A = \text{supp}(\alpha)$, $B = \text{supp}(\beta)$ and $C = \text{supp}(\gamma)$. Observe that $C \vee [X] \leq A \vee B$: indeed, if $(\alpha \circ \beta)(e) = 0$, then $\alpha(e) = 0$, so $e \notin S(\alpha, \beta)$, which implies that $\gamma(e) = (\alpha \circ \beta)(e) = 0$.

We first argue that $\text{res}_{\xi(X)}(\gamma)(x)$ is 0. By construction, it is either $\gamma(x)$ or 1. Suppose it is 1. Then $x \notin \xi|_{\xi(X)}(C|_{\xi(X)}) \cup \Gamma|_{\xi(X)}(C|_{\xi(X)}) = \xi(C \vee [X]) \cup \Gamma(C \vee [X])$. By Proposition 3.1.3, since $C \vee [X] \leq A \vee B$, $x \notin \Gamma(A \vee B) \cup \xi(A \vee B)$. This implies $(\alpha \circ \beta)(x) = 1$, which contradicts $(\alpha \circ \beta)(x) \neq 1$. Thus, $\text{res}_{\xi(X)}(\gamma)(x) = \gamma(x) = 0$.

Let $y \notin S(\text{res}_{\xi(X)}(\alpha), \text{res}_{\xi(X)}(\beta))$. Then $y \notin S(\alpha, \beta)$. Suppose $\text{res}_{\xi(X)}(\alpha \circ \beta)(y) \neq 1$. Then $(\alpha \circ \beta)(y) \neq 1$. This implies, as above, that $\text{res}_{\xi(X)}(\gamma)(y) \neq 1$. Thus, $\text{res}_{\xi(X)}(\gamma)(y) = \gamma(y) = (\alpha \circ \beta)(y)$. By Lemma 4.4.2, $\text{res}_{\xi(X)}(\alpha \circ \beta) = (\alpha \circ \beta)|_{\xi(X)}$, so $\text{res}_{\xi(X)}(\gamma)(y) = \text{res}_{\xi(X)}(\alpha \circ \beta)(y) = (\text{res}_{\xi(X)}(\alpha) \circ \text{res}_{\xi(X)}(\beta))(y)$. \square

The following result identifies the semigroup $\mathcal{G}|_{\xi(X)}$ with a subsemigroup of \mathcal{G} .

Proposition 4.4.4. *Let $(E, \mathcal{F}, \mathcal{G})$ denote an oriented interval greedoid and let $X \in \mathcal{F}$. Then $\text{res}_{\xi(X)}(\beta) \mapsto \alpha \circ \beta$ defines a semigroup isomorphism*

$$\mathcal{G}|_{\xi(X)} \xrightarrow{\cong} \mathcal{G}_{\geq \alpha} = \{\beta \in \mathcal{G} : \beta \geq \alpha\},$$

where α is any covector with $\text{supp}(\alpha) = [X]$.

Proof. Define a map $g : \mathcal{G}_{\geq \alpha} \rightarrow \mathcal{G}|_{\xi(X)}$ by $\beta \mapsto \text{res}_{\xi(X)}(\beta)$. Then g is a semigroup morphism by Proposition 4.2.5. Define a map $f : \mathcal{G}|_{\xi(X)} \rightarrow \mathcal{G}_{\geq \alpha}$ by $f(\text{res}_{\xi(X)}(\beta)) = \alpha \circ \beta$.

We argue that f is well-defined. Suppose $\beta, \gamma \in \mathcal{G}$ with $B = \text{supp}(\beta)$ and $C = \text{supp}(\gamma)$, and suppose $\text{res}_{\xi(X)}(\beta) = \text{res}_{\xi(X)}(\gamma)$. Then the support of $\text{res}_{\xi(X)}(\beta) = \text{res}_{\xi(X)}(\gamma)$ is $B \vee [X] = C \vee [X]$. This implies that $\text{supp}(\alpha \circ \beta) = \text{supp}(\alpha \circ \gamma)$ because $\text{supp}(\alpha) = [X]$. Therefore, to show $\alpha \circ \beta = \alpha \circ \gamma$ it suffices to show that they agree on $\Gamma([X] \vee B)$. Let $x \in \Gamma([X] \vee B)$. Then $x \in \Gamma(X) \cup \xi(X)$ by Proposition 3.1.3. If $x \in \Gamma(X)$, then $(\alpha \circ \beta)(x) = \alpha(x) = (\alpha \circ \gamma)(x)$. So suppose $x \in \xi(X)$. Then $(\alpha \circ \beta)(x) = \beta(x)$ and $(\alpha \circ \gamma)(x) = \gamma(x)$. Moreover, $x \in \Gamma|_{\xi(X)}(B|_{\xi(X)})$ and $x \in$

$\Gamma|_{\xi(X)}(C|_{\xi(X)})$ by Lemma 4.4.1. So $\text{res}_{\xi(X)}(\beta)(x) = \beta(x)$ and $\text{res}_{\xi(X)}(\gamma)(x) = \gamma(x)$. Since $\text{res}_{\xi(X)}(\beta) = \text{res}_{\xi(X)}(\gamma)$, we have $(\alpha \circ \beta) = (\alpha \circ \gamma)$.

Now f is a semigroup morphism since $\alpha \circ \beta \circ \alpha = \alpha \circ \beta$ for all covectors α and β (see Proposition 3.1.4). To complete the proof observe that $f \circ g$ and $g \circ f$ are the identity morphisms of $\mathcal{G}|_{\xi(X)}$ and $\mathcal{G}_{\geq \alpha}$, respectively. \square

5. STRUCTURE OF ORIENTED INTERVAL GREEDOIDS

5.1. \mathcal{G} is a graded poset. The next result generalizes [BLVS⁺93, Lemma 4.1.12] from oriented matroids to oriented interval greedoids.

Lemma 5.1.1. *Let $(E, \mathcal{F}, \mathcal{G})$ denote an oriented interval greedoid. Suppose $\alpha, \beta \in \mathcal{G}$ with $\text{supp}(\alpha) \leq \text{supp}(\beta)$ and $\alpha \not\leq \beta$. Then there exists $\delta \in \mathcal{G}$ such that $\delta < \beta$ and for all $x \notin S(\alpha, \beta)$, if $\beta(x) \neq 1$, then $\delta(x) = \beta(x)$.*

Proof. Let $A = \text{supp}(\alpha)$ and $B = \text{supp}(\beta)$. Suppose the result is not true. Of all $\alpha, \beta \in \mathcal{G}$ that violate the result choose a pair with $|S(\alpha, \beta)|$ minimal. If $S(\alpha, \beta) = \emptyset$, then $\alpha \leq \beta$ by Lemma 3.2.5, contradicting the assumption that $\alpha \not\leq \beta$. Therefore, $S(\alpha, \beta) \neq \emptyset$. Let $y \in S(\alpha, \beta)$. If $(\alpha \circ \beta)(y) = 1$, then $(\beta \circ \alpha)(y) = 1$ and so $\beta(y) = 1$ because $A \leq B$ implies $(\beta \circ \alpha) = \beta$. This contradicts the fact that $y \in S(\alpha, \beta)$. Therefore, $(\alpha \circ \beta)(y) \neq 1$. (OG4) implies there exists $\gamma \in \mathcal{G}$ with $\gamma(y) = 0$ and for all $x \notin S(\alpha, \beta)$, if $\beta(x) = (\beta \circ \alpha)(x) \neq 1$, then $\gamma(x) = (\alpha \circ \beta)(x) = (\beta \circ \alpha)(x) = \beta(x)$.

We argue that $S(\gamma, \beta) \subsetneq S(\alpha, \beta)$. Suppose $e \notin S(\alpha, \beta)$. Then either $\beta(e) = 1$ or $\gamma(e) = \beta(e)$. In both cases $e \notin S(\gamma, \beta)$. Since $y \in S(\alpha, \beta)$ and $y \notin S(\gamma, \beta)$ (because $\gamma(y) = 0$), the inclusion is proper.

Let $C = \text{supp}(\gamma)$. We argue that $C < B$ by showing that $\beta(e) = 0$ implies $\gamma(e) = 0$ and that $B \neq C$. If $\beta(e) = 0$, then $e \notin S(\alpha, \beta)$, so $\beta(e) = 1$ (not possible) or $\gamma(e) = \beta(e) = 0$. Since $\gamma(y) = 0$ and $\beta(y) \in \{+, -\}$, we have $B \neq C$.

We argue that $S(\gamma, \beta) \neq \emptyset$. Suppose $S(\gamma, \beta) = \emptyset$. Then $\gamma < \beta$ by Lemma 3.2.5. Let $\delta \in \mathcal{G}$ denote a coatom in the interval $[\gamma, \beta]$ of the poset \mathcal{G} and let $D = \text{supp}(\delta)$. We will argue that δ satisfies the result, contradicting our assumption that no such δ exists. First note that $\delta < \beta$ by the choice of δ . It remains to show that for all $x \notin S(\alpha, \beta)$, if $\beta(x) \neq 1$, then $\delta(x) = \beta(x)$. Let $x \notin S(\alpha, \beta)$. If $\beta(x) \neq 1$, then $\gamma(x) = \beta(x)$. Since $\gamma(x) \leq \delta(x) \leq \beta(x)$ and $\gamma(x) = \beta(x)$, it follows that $\delta(x) = \beta(x)$. And if $\beta(x) = 1$, then $\gamma(x) \neq 0$, so $\delta(x) \geq \gamma(x) > 0$.

We argued above that $C < B$ and $S(\gamma, \beta) \neq \emptyset$. Hence, $\gamma \not\leq \beta$ by Lemma 3.2.5. Since $\emptyset \neq S(\gamma, \beta) \subsetneq S(\alpha, \beta)$, the minimality of $|S(\alpha, \beta)|$ implies there exists $\delta \in \mathcal{G}$ such that $\delta < \beta$ and for all $x \notin S(\gamma, \beta)$, if $\beta(x) \neq 1$, then $\delta(x) = \beta(x)$. But since $S(\gamma, \beta) \subsetneq S(\alpha, \beta)$, if $x \notin S(\alpha, \beta)$, then $x \notin S(\gamma, \beta)$. Thus, for all $x \notin S(\alpha, \beta)$, if $\beta(x) \neq 1$, then $\delta(x) = \beta(x)$. This contradicts the assumption that no such δ exists for the pair α and β . We have arrived at a contradiction; so the result is true. \square

Example 5.1.2. Figure 8 illustrates Lemma 5.1.1 for the antimatroid corresponding to the convex geometry on three colinear points (Example 2.3.2). \circ

The partial order on the set of all covectors restricts to a partial order on \mathcal{G} . This next result shows that \mathcal{G} is a graded poset and describes the rank function of \mathcal{G} .

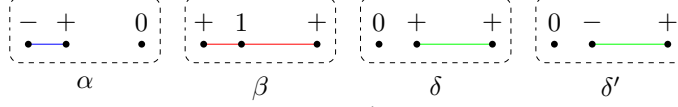


FIGURE 8. The covectors δ and δ' both satisfy the statement of Lemma 5.1.1 for the covectors α and β .

Proposition 5.1.3. *Let \mathcal{G} be an oriented interval greedoid over (E, \mathcal{F}) . Then $\text{supp} : \mathcal{G} \rightarrow \Phi$ is a cover-preserving poset surjection of \mathcal{G} onto Φ satisfying*

$$\text{supp}(\alpha \circ \beta) = \text{supp}(\alpha) \vee \text{supp}(\beta).$$

In particular, \mathcal{G} is graded of rank equal to the rank of Φ . The rank of $\alpha \in \mathcal{G}$ is the rank of $\text{supp}(\alpha) \in \Phi$.

Proof. The identity follows immediately because if $A = \text{supp}(\alpha)$ and $B = \text{supp}(\beta)$, then $\text{supp}(\alpha \circ \beta) = A \vee B$, by definition of the product. The fact that supp is a surjection of posets follows from its definition and axiom (OG1). It remains to show that supp is cover-preserving.

Suppose $\alpha < \beta$. Let $A = \text{supp}(\alpha)$ and $B = \text{supp}(\beta)$. Suppose there exists $C \in \Phi$ such that $A < C < B$. Let $\mathcal{G}' = \mathcal{G}|_{\xi(A)}$. Let α' and β' be the elements of \mathcal{G}' corresponding to α, β .

Since $\text{supp} : \mathcal{G}' \rightarrow [A, \hat{1}]$ is surjective, there exists $\epsilon' \in \mathcal{G}'$ with $\text{supp}(\epsilon') = C$. Let $\gamma' = \alpha' \circ \epsilon'$. Then $\alpha' < \gamma'$ and $\text{supp}(\gamma') = A \vee C = C$. If $\gamma' \leq \beta'$, then $\gamma' = \beta'$, contradicting that $C < B$. Hence, $\gamma' \not\leq \beta'$. By Lemma 5.1.1 there exists $\delta' \in \mathcal{G}'$ such that $\delta' < \beta'$ and for all $x \notin S(\gamma', \beta')$, if $\beta'(x) \neq 1$, then $\delta'(x) = \beta'(x)$. Let $D = \text{supp}(\delta')$. Since $\delta' \in \mathcal{G}'$, $A < D$. Thus $\alpha < \alpha \circ \delta < \alpha \circ \beta = \beta$ (see Proposition 3.1.4), contradicting that $\alpha < \beta$. \square

5.2. Oriented interval greedoids of rank 1. Let $(E, \mathcal{F}, \mathcal{G})$ be an oriented interval greedoid and let Φ be its lattice of flats. The previous result shows that \mathcal{G} is a graded lattice and that its rank is equal to that of Φ . We define the **rank** of $(E, \mathcal{F}, \mathcal{G})$ to be the rank of \mathcal{G} (equivalently, the rank of Φ).

We first make a useful observation about arbitrary oriented interval greedoids.

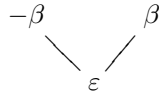
Lemma 5.2.1. *Suppose $(E, \mathcal{F}, \mathcal{G})$ is an oriented interval greedoid. Let $\hat{0}$ be the minimal element of Φ . Then there is a unique element of \mathcal{G} with support $\hat{0}$.*

Proof. By (OG1), there exists $\varepsilon \in \mathcal{G}$ with $\text{supp}(\varepsilon) = \hat{0}$. Then $\Gamma(\text{supp}(\varepsilon)) = \emptyset$, so $\varepsilon(e) \in \{0, 1\}$ for all $e \in E$. Thus, ε is determined by \mathcal{F} , and consequently is the unique element of \mathcal{G} with support $\hat{0}$. \square

We will consistently denote the unique element of \mathcal{G} with support $\hat{0}$ by ε .

The next result describes the oriented interval greedoids of rank 1.

Proposition 5.2.2. *Suppose $(E, \mathcal{F}, \mathcal{G})$ is an oriented interval greedoid of rank 1. Then \mathcal{G} contains exactly three elements, and its Hasse diagram is*



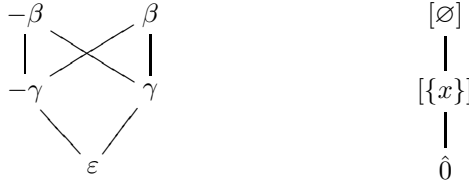
Proof. Since \mathcal{G} has rank 1, Φ contains exactly two elements, its minimal and maximal elements $\hat{0}$ and $\hat{1} = [\emptyset]$, respectively.

By the previous lemma, ε is the unique element of \mathcal{G} with support $\hat{0}$. We now show that there exist exactly two elements in \mathcal{G} of support $\hat{1}$. By (OG1), there exists $\beta \in \mathcal{G}$ such that $\text{supp}(\beta) = \hat{1}$. Then $-\beta \in \mathcal{G}$ by (OG2). Since $\Gamma(\emptyset) \neq \emptyset$, $\beta \neq -\beta$. So \mathcal{G} contains at least two elements of support $\hat{1}$.

Let $\alpha \in \mathcal{G}$, $\alpha \neq \beta$ and $\text{supp}(\alpha) = \hat{1}$. Let $y \in \Gamma(\emptyset)$, $y \notin S(\alpha, \beta)$. Then let δ be the vector guaranteed by Lemma 5.1.1. Then $\delta(y) = \beta(y)$. But $\delta = \varepsilon$, so this is impossible. It follows that $S(\alpha, \beta) = \Gamma(\emptyset)$; in other words, $\alpha = -\beta$. \square

5.3. Oriented interval greedoids of rank 2.

Proposition 5.3.1. *Suppose $(E, \mathcal{F}, \mathcal{G})$ is an oriented interval greedoid of rank 2. Then \mathcal{G} is isomorphic to the semigroup of covectors of an oriented matroid of rank 2, or the Hasse diagrams of \mathcal{G} and Φ are, respectively, the following two posets.*



Proof. There are two cases to consider.

Case 5.3.1.1. Suppose Φ contains at least two coatoms. By Proposition 4.3.1, the restriction $\mathcal{G}|_{\Gamma(\emptyset)}$ is an oriented matroid. The map $C \mapsto [Y]$ for any $Y \in C$ embeds $\Phi|_{\Gamma(\emptyset)}$ into the interval $[[X], \hat{1}]$ of Φ , where X is the maximal among the feasible sets contained in $\Gamma(\emptyset)$ (see §4.2.1). Since every coatom of Φ is of the form $[\{x\}]$ for some $x \in \Gamma(\emptyset)$, there is a bijection between the coatoms of Φ and those of $\Phi|_{\Gamma(\emptyset)}$. Therefore, $[X] = \hat{0}$, so $\Phi|_{\Gamma(\emptyset)} \cong \Phi$. So $\mathcal{G}|_{\Gamma(\emptyset)}$ is a rank 2 oriented matroid. We argue that the map $\gamma \mapsto \gamma|_{\Gamma(\emptyset)}$ is an isomorphism $\mathcal{G} \cong \mathcal{G}|_{\Gamma(\emptyset)}$. By Proposition 4.3.1 we need only show that this is an injection.

Let $\gamma, \gamma' \in \mathcal{G}$ and suppose $\gamma|_{\Gamma(\emptyset)} = \gamma'|_{\Gamma(\emptyset)}$. Then $\text{supp}(\gamma)|_{\Gamma(\emptyset)} = \text{supp}(\gamma')|_{\Gamma(\emptyset)}$. Since $\Phi|_{\Gamma(\emptyset)} \cong \Phi$, it follows that $\text{supp}(\gamma) = \text{supp}(\gamma')$. This implies that $\gamma(x)$ is 0 or 1 if and only if $\gamma'(x)$ is 0 or 1, respectively. Let $C = \text{supp}(\gamma) = \text{supp}(\gamma')$.

If $C = \hat{0}$, then $\gamma = \gamma'$ since there is a unique element of \mathcal{G} with support $\hat{0}$. If $C = \hat{1} = [\emptyset]$, then $\gamma|_{\Gamma(\emptyset)} = \gamma'|_{\Gamma(\emptyset)}$ implies that $\gamma = \gamma'$ since they agree on $\Gamma(\emptyset)$.

Let $C \succ \hat{0}$. Suppose $\gamma \neq \gamma'$. Arguing as in the end of Proposition 5.2.2, we conclude $\Gamma(C) = S(\gamma, \gamma')$. Since γ and γ' agree on $\Gamma(\emptyset)$ and disagree on $\Gamma(C)$, it follows $\gamma|_{\Gamma(\emptyset)} = \gamma'|_{\Gamma(\emptyset)}$ takes values in $\{0, 1\}$. Thus, $C|_{\Gamma(\emptyset)} = \hat{0}$, which implies $C = \hat{0}$, contradicting that $C \succ \hat{0}$. Thus, $\gamma = \gamma'$.

Case 5.3.1.2. Suppose Φ contains exactly one coatom. Then $[X]$ is this coatom, so $X = \{x\}$ for some $x \in E$. By Lemma 5.2.1, ε is the unique element of \mathcal{G} with support $\hat{0}$.

By (OG1), there exists $\gamma \in \mathcal{G}$ such that $\text{supp}(\gamma) = [X]$. By arguing as in Proposition 5.2.2, we conclude that $\gamma \neq -\gamma$ and that if $\nu \in \mathcal{G}$ with $\text{supp}(\nu) = [X]$, then $\nu = \gamma$ or $\nu = -\gamma$. Hence, there are exactly two elements in \mathcal{G} of support $[X]$.

By (OG1), there exists $\beta \in \mathcal{G}$ such that $\text{supp}(\beta) = [\emptyset]$. By (OG2) $-\beta \in \mathcal{G}$. As above, we have $\beta \neq -\beta$. $\Gamma(\emptyset) \cap \Gamma(X) = \emptyset$, since if $y \in \Gamma(\emptyset) \cap \Gamma(X)$, then $y \notin \xi(X)$, so $[y] \neq [x]$, contradicting our assumption that Φ has only one coatom. Thus $\gamma, -\gamma < \beta, -\beta$.

Let $\nu \in \mathcal{G}$ such that $\text{supp}(\nu) = \hat{1}$. By arguing as before (using Lemma 5.1.1), it follows that $\nu = \beta$ or $\nu = -\beta$. \square

5.4. Intervals of length two. Let $\hat{\mathcal{G}}$ denote the poset obtained from \mathcal{G} by adjoining a maximal element $\hat{1}$. We prove that all intervals of length two in $\hat{\mathcal{G}}$ contain exactly four elements.

Proposition 5.4.1. *Suppose \mathcal{G} is an oriented interval greedoid. Then all intervals in $\hat{\mathcal{G}}$ of length two contain exactly four elements.*

Proof. Let $\alpha, \beta, \gamma \in \hat{\mathcal{G}}$ such that $\alpha < \gamma < \beta$. The case where $\beta = \hat{1}$ was proved in Proposition 5.2.2, so suppose $\beta \in \mathcal{G}$. Let $A = \text{supp}(\alpha)$, $B = \text{supp}(\beta)$ and $C = \text{supp}(\gamma)$. By Proposition 5.1.3, $A < C < B$ in Φ .

Let $\text{supp}(\alpha) = [X]$ for some $X \in \mathcal{F}$. By Proposition 4.4.4 and Proposition 3.1.4, $\mathcal{G}|_{\xi(X)} \cong \mathcal{G}_{\geq \alpha}$ (as posets), so $\{\delta \in \mathcal{G} : \alpha < \delta < \beta\} \cong \{\delta \in \mathcal{G}|_{\xi(X)} : \alpha|_{\xi(X)} < \delta < \beta|_{\xi(X)}\}$. Thus, by passing to $\mathcal{G}|_{\xi(X)}$ we can suppose that $A = \hat{0}$.

Let $\text{supp}(\beta) = [Y]$ for some $Y \in \mathcal{F}$. By Proposition 4.1.4, $\mathcal{G}/Y \cong \mathcal{G}_{\leq [Y]} = \{\nu \in \mathcal{G} : \text{supp}(\nu) \leq [Y]\}$. Since $\Phi/Y \cong [\hat{0}, [Y]] \subseteq \Phi$ (Proposition 4.1.1), by passing to \mathcal{G}/Y , we can suppose that $B = \hat{1}$, and therefore that Φ is a lattice of rank 2.

Proposition 5.3.1 classified the oriented interval greedoids of rank 2 as being either an oriented matroid of rank 2 or having the Hasse diagram shown in the statement of Proposition 5.3.1. For the latter situation a quick inspection of the given poset establishes the result. And for the former situation, it is well-known that this result holds for oriented matroids ([BLVS⁺93, Theorem 4.1.14]). \square

5.5. The Underlying Oriented Matroid. Let $(E, \mathcal{F}, \mathcal{G})$ be an oriented interval greedoid. The top element in the poset of flats Φ is $[\emptyset]$, and by Proposition 3.1.3 it follows that $\Gamma(\emptyset) \subseteq \Gamma(A) \cup \xi(A)$ for any flat $A \in \Phi$. This implies that $\alpha(x) \in \{0, +, -\}$ for any $\alpha \in \mathcal{G}$ and any $x \in \Gamma(\emptyset)$. Moreover, $\Gamma(\emptyset)$ is the largest subset of E with this property: if α is maximal in \mathcal{G} , then $\text{supp}(\alpha) = [\emptyset]$ and $\alpha(x) = 1$ if and only if $x \notin \Gamma(\emptyset)$. This observation implies that the restriction to $\Gamma(\emptyset)$ produces an oriented interval greedoid whose covectors take values in $\{0, +, -\}$. Thus, $\mathcal{G}|_{\Gamma(\emptyset)}$ is an oriented matroid. Alternatively, one can note that the restriction $(\Gamma(\emptyset), \mathcal{F}|_{\Gamma(\emptyset)})$ is a matroid and appeal to Theorem 3.4.1.

Definition 5.5.1. Let \mathcal{G} denote an oriented interval greedoid over (E, \mathcal{F}) . The *underlying oriented matroid* of \mathcal{G} is $\bar{\mathcal{G}} = \mathcal{G}|_{\Gamma(\emptyset)}$.

The lattice of flats $\bar{\Phi}$ of $\bar{\mathcal{G}}$ is a geometric lattice because $\bar{\mathcal{G}}$ is an oriented matroid. Moreover, it is isomorphic to the sublattice of Φ generated by all the coatoms.

5.6. The Tope Graph. A **tope** of an oriented interval greedoid $(E, \mathcal{F}, \mathcal{G})$ is a covector that is maximal in \mathcal{G} with respect to the partial order on covectors. Alternatively, topes are covectors whose support is $\hat{1} = [\emptyset]$. A **subtope** of \mathcal{G} is a covector in \mathcal{G} that is covered by some tope. From Proposition 5.4.1 it follows that every subtope is covered by exactly two different topes. Two topes are said to be **adjacent** if there exists a subtope that is covered by both topes.

The **tope graph** $\mathsf{T}(\mathcal{G})$, or just T , of \mathcal{G} is the graph with one vertex for each tope of \mathcal{G} and an edge between two vertices if the corresponding topes are adjacent.

Lemma 5.6.1. *Suppose \mathcal{G} is an oriented interval greedoid. Then the tope graph of \mathcal{G} is isomorphic to the tope graph of the underlying oriented matroid $\overline{\mathcal{G}}$ of \mathcal{G} .*

Proof. First we will show that topes of \mathcal{G} are in one-to-one correspondence with the topes of $\overline{\mathcal{G}}$. Suppose α is a tope in \mathcal{G} . Then $\text{supp}(\alpha) = [\emptyset] = \{\emptyset\}$, and so $\text{supp}|_{\Gamma(\emptyset)}(\text{res}_{\Gamma(\emptyset)}(\alpha)) = \{\emptyset\} = \hat{1} \in \Phi|_{\Gamma(\emptyset)}$. Thus, $\text{res}_{\Gamma(\emptyset)}(\alpha)$ is a tope of $\overline{\mathcal{G}}$.

Conversely, suppose $\text{res}_{\Gamma(\emptyset)}(\alpha)$ is a tope of $\overline{\mathcal{G}}$. Then $\text{supp}|_{\Gamma(\emptyset)}(\text{res}_{\Gamma(\emptyset)}(\alpha)) = [\emptyset]|_{\Gamma(\emptyset)} = \{\emptyset\}$. Let $A = \text{supp}(\alpha)$. Then $A|_{\Gamma(\emptyset)} = \{\emptyset\}$, so \emptyset is maximal among the feasible sets contained in $\Gamma(\emptyset) \cap \xi(A)$. This implies that $\Gamma(\emptyset) \cap \xi(A) = \emptyset$ (if $x \in \Gamma(\emptyset) \cap \xi(A)$, then $\{x\} \in \mathcal{F}$ because $x \in \Gamma(\emptyset)$, contradicting that maximality of \emptyset). If $A \neq [\emptyset]$, then $A \leq [\{y\}]$ for some $y \in \Gamma(\emptyset)$. Hence, $y \in \Gamma(\emptyset) \cap \xi(A)$, contradicting that $\Gamma(\emptyset) \cap \xi(A) = \emptyset$. Thus, $A = [\emptyset]$. So α is a tope of \mathcal{G} .

Let α and β be topes in \mathcal{G} and suppose $\text{res}_{\Gamma(\emptyset)}(\alpha) = \text{res}_{\Gamma(\emptyset)}(\beta)$. We show that $\alpha = \beta$ by showing that they agree on $\Gamma(\text{supp}(\alpha)) = \Gamma(\text{supp}(\beta)) = \Gamma(\emptyset)$. Since $\text{supp}(\alpha) = \text{supp}(\beta) = [\emptyset]$, we have $\Gamma(\emptyset) = \Gamma|_{\Gamma(\emptyset)}(\emptyset) = \Gamma|_{\Gamma(\emptyset)}([\emptyset]|_{\Gamma(\emptyset)})$. Hence, $\text{res}_{\Gamma(\emptyset)}(\alpha)(w) = \alpha(w)$ and $\text{res}_{\Gamma(\emptyset)}(\beta)(w) = \beta(w)$ for all $w \in \Gamma(\emptyset)$. It follows that $\alpha(w) = \beta(w)$ for all $w \in \Gamma(\emptyset)$. This establishes the one-to-one correspondence.

Suppose $\alpha, \beta \in \mathcal{G}$ are two adjacent topes and let $\gamma \in \mathcal{G}$ with $\gamma \triangleleft \alpha, \beta$. Then $\text{supp}(\gamma) \triangleleft \text{supp}(\alpha) = \text{supp}(\beta) = [\emptyset]$. Since $\text{res}_{\Gamma(\emptyset)}$ is a semigroup morphism, it follows that $\text{res}_{\Gamma(\emptyset)}(\gamma) \leq \text{res}_{\Gamma(\emptyset)}(\alpha)$. We cannot have equality since this would imply that both are topes of $\mathcal{G}|_{\Gamma(\emptyset)}$, contradicting that γ is not a tope. We have $\text{supp}(\gamma) = [\{y\}]$ for some $y \in \Gamma(\emptyset)$ since all coatoms of Φ are of this form. Hence, $\text{res}_{\Gamma(\emptyset)}(\gamma) = [\{y\}]|_{\Gamma(\emptyset)} \triangleleft [\emptyset]|_{\Gamma(\emptyset)}$. Since $\text{supp}|_{\Gamma(\emptyset)}$ is cover-preserving, it follows that $\text{res}_{\Gamma(\emptyset)}(\gamma) \triangleleft \text{res}_{\Gamma(\emptyset)}(\alpha)$. Similarly, $\text{res}_{\Gamma(\emptyset)}(\gamma) \triangleleft \text{res}_{\Gamma(\emptyset)}(\beta)$. So $\text{res}_{\Gamma(\emptyset)}(\alpha)$ and $\text{res}_{\Gamma(\emptyset)}(\beta)$ are adjacent topes.

Let $\text{res}_{\Gamma(\emptyset)}(\alpha), \text{res}_{\Gamma(\emptyset)}(\beta) \in \mathcal{G}|_{\Gamma(\emptyset)}$ be adjacent topes and let $\text{res}_{\Gamma(\emptyset)}(\gamma) \in \mathcal{G}|_{\Gamma(\emptyset)}$ with $\text{res}_{\Gamma(\emptyset)}(\gamma) \in \mathcal{G}$ with $\text{res}_{\Gamma(\emptyset)}(\gamma) \triangleleft \text{res}_{\Gamma(\emptyset)}(\alpha), \text{res}_{\Gamma(\emptyset)}(\beta)$. Since $\text{res}_{\Gamma(\emptyset)}(\gamma \circ \alpha) = \text{res}_{\Gamma(\emptyset)}(\gamma) \circ \text{res}_{\Gamma(\emptyset)}(\alpha) = \text{res}_{\Gamma(\emptyset)}(\alpha)$ and since $\gamma \circ \alpha$ and α are both topes, we have $\alpha = \gamma \circ \alpha$. So $\gamma \triangleleft \alpha$. To show that $\gamma \triangleleft \alpha$, it suffices to show that $\text{supp}(\gamma) \triangleleft [\emptyset]$. Let $C = \text{supp}(\gamma)$. If C is not covered by $[\emptyset]$, then $C \leq [\{x, y\}]$ for some $x, y \in \Gamma(\emptyset)$, $x \neq y$. Thus, $\{x, y\} \subseteq \Gamma(\emptyset) \cap \xi(C)$. Let $Y \supseteq \{x, y\}$ be maximal among the feasible sets contained in $\xi(C) \cap \Gamma(\emptyset)$. By definition, $\text{supp}|_{\Gamma(\emptyset)}(\text{res}_{\Gamma(\emptyset)}(\gamma)) = C|_{\Gamma(\emptyset)}$ is the flat containing Y . Since $|Y| > 2$, it follows that $\text{supp}|_{\Gamma(\emptyset)}(\text{res}_{\Gamma(\emptyset)}(\gamma))$ is not a coatom of $\Phi|_{\Gamma(\emptyset)}$, contradicting that it is. Hence, $\gamma \triangleleft \alpha$. Similarly, $\gamma \triangleleft \beta$. Therefore, α and β are adjacent topes. \square

6. CW-SPHERES FROM ORIENTED INTERVAL GREEDOIDS

6.1. CW-spheres. The Sphericity Theorem is an important result for oriented matroids which asserts that there is a certain regular CW-sphere associated to any oriented matroid, whose cells correspond to the covectors of the oriented matroid. It is originally due to Folkman and Lawrence [FL78]; see also [BLVS⁺93, Theorem 4.3.3]. In this section and the next, we will prove the corresponding result for oriented interval greedoids.

We recall some topological definitions, following [BLVS⁺93, Section 4.7].

A **ball** in a topological space homeomorphic to the usual d -dimensional ball, for some nonnegative integer d .

A **regular cell complex** Δ is a finite set of balls in a Hausdorff topological space $|\Delta| = \bigcup_{\sigma \in \Delta} \sigma$ with the properties that:

- The interiors of the balls $\sigma \in \Delta$ partition $|\Delta|$.
- For each $\sigma \in \Delta$, the boundary of σ is the union of some elements $\tau \in \Delta$.

This definition of a regular cell complex is (non-trivially) equivalent to the usual definition of a regular CW-complex. (See [BLVS⁺93, Section 4.7].)

A cell complex Δ is called a **regular CW-sphere** if its geometric realization $|\Delta|$ is homeomorphic to a sphere.

The **face poset** of a cell complex is the poset structure on the cells of Δ , ordered by containment. The **augmented face poset** of a cell complex is the face poset with a maximal element $\hat{1}$ adjoined.

We can now state our main theorem for this section more precisely.

Theorem 6.1.1. *For $(E, \mathcal{F}, \mathcal{G})$ an oriented interval greedoid, $\hat{\mathcal{G}}$ is isomorphic to the augmented face poset of a regular CW-sphere.*

The **order complex** of a bounded poset P is the simplicial complex consisting of chains in $P \setminus \{\hat{0}, \hat{1}\}$. Taking a barycentric subdivision of the CW-sphere in the previous theorem, we obtain the following.

Corollary 6.1.2. *The order complex of $\hat{\mathcal{G}}$ is a simplicial sphere.*

Proof of Theorem 6.1.1. As in the proof of the Sphericity Theorem in [BLVS⁺93], the main technical tool required in the proof is the notion of *recursive coatom ordering*.

A graded, bounded poset P is said to have a **recursive coatom ordering** if it is either of rank 1, or if there is a linear ordering on its coatoms, q_1, \dots, q_r which satisfies:

- (i) $[\hat{0}, q_i]$ admits a recursive coatom ordering in which the coatoms of $[\hat{0}, q_i]$ which lie below some q_j with $j < i$, come first;
- (ii) any element lying below q_i and also below some q_j for $j < i$, necessarily lies below a coatom of $[\hat{0}, q_i]$ which lies below some q_k with $k < i$.

This concept is dual to the condition of having a recursive atom ordering, which goes back to [BW83]. The concept has been extended to non-graded posets [BW96], but we shall not need that here.

The fundamental technical result is the following lemma, whose proof we defer to the next section.

Lemma 6.1.3. $\hat{\mathcal{G}}$ admits a recursive coatom ordering.

A poset is called *thin* if all intervals of length 2 have cardinality four. By Proposition 5.4.1, we know that $\hat{\mathcal{G}}$ is thin. The following theorem completes our proof.

Theorem 6.1.4 ([Bjö84], [BLVS⁺93, Theorem 4.7.24]). *P is isomorphic to the face poset of a shellable regular cell decomposition of the sphere iff P is thin and admits a recursive coatom ordering.*

(We shall not discuss the significance of the “shellable” in the above theorem; the interested reader is directed to [BLVS⁺93].) \square

We now turn to the proof of the corollary.

Proof of Corollary 6.1.2. The order complex of the augmented face poset of a regular cell complex Δ is homeomorphic to $|\Delta|$ [BLVS⁺93, Proposition 4.7.8]. (In fact, the order complex should be thought of as the barycentric subdivision of the regular cell complex.) The corollary follows. \square

6.2. A recursive coatom ordering for $\hat{\mathcal{G}}$. This section is devoted to the proof of Lemma 6.1.3, which asserts that $\hat{\mathcal{G}}$ has a recursive coatom ordering.

If one chooses a particular tope α of \mathcal{G} then there is a natural poset structure on the topes with respect to which α is the minimum element and $-\alpha$ is the maximum element, and the Hasse diagram is (a suitable orientation of) the tope graph. This poset is called $\mathcal{T}(\mathcal{G}, \alpha)$. (Since the topes of \mathcal{G} are identified with the topes of $\bar{\mathcal{G}}$, this follows from the analogous statements for oriented matroids; see [BLVS⁺93, Section 4.2].)

Let α be a tope of \mathcal{G} . Consider a maximal chain $\bar{\beta}$ in $\mathcal{T}(\mathcal{G}, \alpha)$, say $\alpha = \beta_0 < \dots < \beta_r = -\alpha$. Choose γ_i to be a common facet of β_{i-1} and β_i . Let $G_i = \text{supp}(\gamma_i)$. The G_i are distinct and include all the coatoms of Φ . Thus, $\bar{\beta}$ induces a linear order on the coatoms of Φ . However (unlike the situation for oriented matroids) this does not immediately yield a linear order on the coatoms of $[\varepsilon, \alpha]$, because there may be more than one coatom with the same support.

For $1 \leq i \leq r$, let \mathcal{G}_i be the oriented matroid obtained by contracting \mathcal{G} to G_i . Consider the tope poset $\mathcal{T}(\mathcal{G}_i, \gamma_i)$.

Let Δ be the set of facets of α . Let Δ_i be the set of facets of α whose support is G_i . (This set could be empty.)

A linear extension of $\mathcal{T}(\mathcal{G}_i, \gamma_i)$ will be called *adapted to α* if it contains in order:

- (1) first, the topes of \mathcal{G}_i that lie on the same side as γ_i of some G_j for $j < i$,
- (2) then, the topes that are facets of α ,
- (3) finally, the remaining topes of \mathcal{G}_i .

We will need the following lemma:

Lemma 6.2.1. $\mathcal{T}(\mathcal{G}_i, \gamma_i)$ admits a linear extension adapted to α .

Proof. It is certainly possible to define a linear extension of $\mathcal{T}(\mathcal{G}_i, \gamma_i)$ which begins with the elements (1) above, since they form a lower order ideal in $\mathcal{T}(\mathcal{G}_i, \gamma_i)$. In order to be able to construct a linear extension such that the next elements are those from (2) above, we need to show that any tope below a tope from (2) not in (2), is contained in (1). If δ is a tope of Δ_i which is a facet of α , and ϵ is a tope lying below δ which is not a facet of α , it must be separated from α by some G_j with $j < i$, which shows that ϵ is in (1). Thus the linear extension, whose beginning was already described, can be continued with the set of facets of α , followed by the remaining topes of \mathcal{G}_i . \square

A linear order on Δ will be said to be *compatible* with $\bar{\beta}$ if

- (1) the elements of Δ are arranged first of all in increasing order by support (so Δ_1 comes first, then Δ_2 , etc.),
- (2) the elements of Δ_i are arranged according to a linear order on $\mathcal{T}(\mathcal{G}_i, \gamma_i)$ which is adapted to α .

Now we will prove the following:

Proposition 6.2.2. (1) *For a tope α in \mathcal{G} , and a maximal chain $\bar{\beta}$ in $\mathcal{T}(\mathcal{G}, \alpha)$, any order on the coatoms of $[\varepsilon, \alpha]$ compatible with $\bar{\beta}$ is a recursive coatom order.*
 (2) *For a tope α in \mathcal{G} , any linear extension of $\mathcal{T}(\mathcal{G}, \alpha)$ is a recursive coatom ordering for $\hat{\mathcal{G}}$.*

Proof. The proof will be by induction on the rank of \mathcal{G} . The base case, when the rank of \mathcal{G} is 1, is trivial. We will assume that (1) and (2) hold for oriented interval greedoids of rank less than n ; we will prove (1) for oriented interval greedoids of rank n , and then make use of (1) to prove (2) for oriented interval greedoids of rank n .

Proof of (1). Pick a coatom order for $[\varepsilon, \alpha]$ which is compatible with $\bar{\beta}$. As part of this, we are given γ_i a common facet of β_{i-1} and β_i . Let G_i be the support of γ_i . Let Δ_i be the coatoms of α with support G_i . As part of our coatom order for $[\varepsilon, \alpha]$, we are given a linear order on Δ_i which is the restriction of a linear extension of $\mathcal{T}(\mathcal{G}_i, \gamma_i)$ adapted to α . Fix such a linear extension.

Let $\delta \in \Delta_i$ be a coatom of $[\varepsilon, \alpha]$. We must define a coatom order for $[\varepsilon, \delta]$. Using our chosen linear extension of $\mathcal{T}(\mathcal{G}_i, \gamma_i)$, we can apply (2) to $\hat{\mathcal{G}}_i$, obtaining a recursive coatom order for $[\varepsilon, \delta]$. We must show that this order satisfies the necessary conditions.

Now, δ is a coatom of two different posets, $[\varepsilon, \alpha]$ and $\hat{\mathcal{G}}_i$. Let X be the set of coatoms of $[\varepsilon, \alpha]$ which precede δ with respect to the coatom order on $[\varepsilon, \alpha]$, and let Y be the set of coatoms of $\hat{\mathcal{G}}_i$ which precede δ with respect to the fixed linear extension of $\mathcal{T}(\mathcal{G}_i, \gamma_i)$. Let \tilde{X} be the coatoms of $[\varepsilon, \delta]$ lying below an element of X , and let \tilde{Y} be the coatoms of $[\varepsilon, \delta]$ lying below an element of Y . We will now show that \tilde{X} and \tilde{Y} coincide.

Let ϵ be a coatom of $[\varepsilon, \delta]$. By Proposition 4.4.4, $\mathcal{G}_{\geq \epsilon}$ is itself an oriented greedoid, so we may assume that $\epsilon = \varepsilon$, or, in other words, that \mathcal{G} is rank 2. By

Proposition 5.3.1, we know that \mathcal{G} is either isomorphic to a rank 2 oriented matroid, or else it is of the special form described in that Proposition. In either case, it is straightforward to check that $\epsilon \in \check{X}$ iff $\epsilon \in \check{Y}$.

Since we know property (i) of recursive coatom orders holds for our fixed linear extension of $\mathcal{T}(\mathcal{G}_i, \gamma_i)$, property (i) also follows for our coatom ordering on $[\epsilon, \alpha]$.

Next, we check property (ii). Let $\epsilon \in [\epsilon, \delta]$, which lies under some element $\zeta \in X$. We must show that it also lies below some element of \check{X} .

Again, by restricting, we may assume that $\epsilon = \epsilon$. The fact that ϵ lies under an element of X implies, in particular, that X is non-empty, and thus that δ is not the first coatom in our coatom order on $[\epsilon, \alpha]$. We will now show that Y is non-empty. If δ is not the first coatom with support G_i in our recursive coatom order on $[\epsilon, \alpha]$ then this is clear. So suppose that δ is the first coatom with support G_i in our recursive coatom order. Since δ is not the first coatom overall, it must be that $i > 1$. Therefore γ_i is not a facet of α , so $\gamma_i \in Y$.

Now, since we have assumed that $\epsilon = \epsilon$, the fact that Y is non-empty means that there are elements of Y lying over ϵ . Therefore, by property (ii) for the fixed linear extension of $\mathcal{T}(\mathcal{G}_i, \gamma_i)$, we know that there are elements of \check{Y} lying over ϵ . Since $\check{Y} = \check{X}$, we are done.

Proof of (2), assuming (1). Pick a linear extension of $\mathcal{T}(\mathcal{G}, \alpha)$. For each coatom δ of $\hat{\mathcal{G}}$, pick a maximal chain $\bar{\beta}$ in $\mathcal{T}(\mathcal{G}, \delta)$ which includes α . Then we claim that any linear order on the coatoms of $[\epsilon, \delta]$, compatible with $\bar{\beta}$, satisfies the necessary conditions. First of all, it is a recursive coatom order by (1).

Second, define Q_δ to be the set of coatoms of $[\epsilon, \delta]$ which also lie under some ξ preceding δ in the linear extension of $\mathcal{T}(\mathcal{G}, \alpha)$. The coatoms of Q_δ precede the other coatoms of δ in any order compatible with $\bar{\beta}$. (In fact, for this, it suffices to know that an order compatible with $\bar{\beta}$ agrees with the order induced by $\bar{\beta}$ on the coatoms of $[\hat{0}, \text{supp}(\delta)]$.) This proves (i).

Thirdly, we check that

$$\bigcup_{\zeta \in Q_\delta} [\epsilon, \zeta] = [\epsilon, \delta] \cap \bigcup_{\xi \text{ preceding } \delta} [\epsilon, \xi].$$

The containment of the lefthand side in the righthandside is obvious. For the opposite inclusion, let $\epsilon \in [\epsilon, \delta] \cap [\epsilon, \xi]$ for some ξ preceding δ . The topes of \mathcal{G} that contain ϵ are exactly the topes of $\bar{\mathcal{G}}$ that contain $\epsilon|_{\Gamma(\emptyset)}$. By [BLVS⁺93, Lemma 4.2.12], this is an interval I in $\mathcal{T}(\bar{\mathcal{G}}, \alpha)$. Since ϵ is contained in some ξ preceding δ , we know that δ is not the minimum element of the interval. Let ρ be covered by δ in I . Since ρ lies below δ in $\mathcal{T}(\mathcal{G}, \alpha)$, it precedes δ in the linear extension of $\mathcal{T}(\mathcal{G}, \alpha)$. Since ρ is in I , $\epsilon \in [\hat{0}, \rho]$. Finally, since ρ and δ are adjacent topes, they have a common subtope σ in $\bar{\mathcal{G}}$. Since, in $\hat{\mathcal{G}}$, ρ and δ lie over $\epsilon|_{\Gamma(\emptyset)}$, σ lies over $\epsilon|_{\Gamma(\emptyset)}$. Thus $\text{supp}(\sigma)$ lies over $\text{supp}(\epsilon)$.

Let ϕ be covector of \mathcal{G} , such that $\phi|_{\Gamma(\emptyset)} = \sigma$. Now consider $\epsilon \circ \phi$. This lies over ϵ , and its support is $\text{supp}(\epsilon) \vee \text{supp}(\phi) = \text{supp}(\phi)$. Since, in $\hat{\mathcal{G}}$, ϕ and ϵ lie below both δ and ρ , the same is true of $\epsilon \circ \phi$, and we are done: we can take $\epsilon \circ \phi$ as the common coatom of $[\epsilon, \delta]$ and $[\epsilon, \rho]$ lying over ϵ . \square

6.3. Face Enumeration. Here, we prove formulas counting chains in an oriented interval greedoid \mathcal{G} . These results generalize results for oriented matroids [BLVS⁺93, Proposition 4.6.2] and for oriented antimatroids [BHP08].

Let P be a poset. Recall that the *Möbius function* of P , denoted μ_P , is the unique function from pairs (x, y) with $x \leq y$ in P to \mathbb{Z} , such that:

- $\mu_P(x, x) = 1$.
- For $x < y$, $\sum_{x \leq z \leq y} \mu_P(y, z) = 0$.

Theorem 6.3.1. *Let $(E, \mathcal{F}, \mathcal{G})$ be an oriented interval greedoid. Let $A_1 > \dots > A_{k+1} = \hat{0}$ be a chain of flats in Φ . Then:*

$$|\text{supp}^{-1}(A_1, \dots, A_{k+1})| = \prod_{i=1}^k \sum_{B \in [A_{i+1}, A_i]} |\mu_\Phi(B, A_i)|,$$

where μ_Φ is the Möbius function of Φ .

First, we state and prove the following special case, which generalizes [BLVS⁺93, Theorem 4.6.1].

Proposition 6.3.2. *Let $(E, \mathcal{F}, \mathcal{G})$ be an oriented interval greedoid. Then the number of topes of \mathcal{G} is:*

$$\sum_{B \in \Phi} |\mu_\Phi(B, \hat{1})|.$$

Proof. One could adapt the proof for oriented matroids to this setting, thus reproving the result for oriented matroids, but we prefer to assume the result if \mathcal{G} is an oriented matroid; this is [BLVS⁺93][Theorem 4.6.1].

Recall that the topes of \mathcal{G} are the same as those of $\overline{\mathcal{G}}$. Applying the proposition to $\overline{\mathcal{G}}$, and writing $\overline{\Phi}$ for $\Phi(\overline{\mathcal{G}})$, we need now only show that:

$$(6.1) \quad \sum_{B \in \Phi} |\mu_\Phi(B, \hat{1})| = \sum_{B \in \overline{\Phi}} |\mu_{\overline{\Phi}}(B, \hat{1})|.$$

Consider the order-preserving map $i : \overline{\Phi} \rightarrow \Phi$ defined by $i([X]) = [X]$, as discussed in §4.2.1. We prove a few more properties of it here.

Lemma 6.3.3. (1) *i is a poset isomorphism onto its image.*
 (2) *For $A, B \in \overline{\Phi}$, we have $i(A \wedge B) = i(A) \wedge i(B)$.*

Proof. (1) Proposition 4.2.3 provides a restriction map from Φ to $\overline{\Phi}$ defined by $A|_{\Gamma(\emptyset)} = \mu|_{\Gamma(\emptyset)}(\xi(A) \cap \Gamma(\emptyset))$, which is order-preserving. Since $i(A)|_{\Gamma(\emptyset)} = A$, we know i is a poset isomorphism onto its image.

(2) Let $A, B \in \overline{\Phi}$. Let $C = i(A) \wedge i(B)$, and let $D = i(C|_{\Gamma(\emptyset)})$. It is immediate that $D \geq C$. However, we know $D \leq i(A)$ and $D \leq i(B)$, so $D = C$. This implies that C is in the image of i , so, by (1), $C = i(A \wedge B)$. \square

Thanks to Lemma 6.3.3 (1), we can identify $\overline{\Phi}$ as a subposet of Φ .

Let $x \in E$ such that $\{x\} \in \mathcal{F}$. Then, by definition, $x \in \Gamma(\emptyset)$. It follows that every coatom of Φ is in $\overline{\Phi}$. Further, since $\overline{\Phi}$ is a geometric lattice, every element of $\overline{\Phi}$ can be written as a meet (in $\overline{\Phi}$) of coatoms. Thanks to Lemma 6.3.3 (2), it

follows that $\overline{\Phi}$ consists exactly of those elements of Φ that can be written as a meet of coatoms in Φ .

Lemma 6.3.4. (1) If $A \in \Phi \setminus \overline{\Phi}$, then $\mu_{\Phi}(A, \hat{1}) = 0$.
 (2) If $A \in \overline{\Phi}$, then $\mu_{\Phi}(A, \hat{1}) = \mu_{\overline{\Phi}}(A, \hat{1})$.

Proof. (1) Since $A \notin \overline{\Phi}$, A cannot be expressed as a meet of coatoms of Φ . It follows that the meet of the coatoms of $[A, \hat{1}]$ is strictly greater than A . The Crosscut Theorem (see [Bjö95]) now implies $\mu_{\Phi}(A, \hat{1}) = 0$.

(2) We induct on the corank of A . The statement is obvious for $A = \hat{1}$. For A of positive corank, we use the formula:

$$\mu_{\Phi}(A, \hat{1}) = - \sum_{A < B \in \Phi} \mu_{\Phi}(B, \hat{1}).$$

Now we observe that, by (1), only the terms with $B \in \overline{\Phi}$ contribute. By induction, these terms agree with $\mu_{\overline{\Phi}}(B, \hat{1})$, which proves the result. \square

(6.1) is now obvious, and the proposition follows. \square

Proof of Theorem 6.3.1. The proof goes exactly as in the oriented matroid case, now that the preparations have been made. $|\text{supp}^{-1}(A_k)|$ is the number of topes of \mathcal{G}/A_k , which is $\sum_{A_k \geq B} |\mu(B, A_k)|$, and then the rest of the chain lies in $\mathcal{G}|_{\xi(A_k)}$, which accounts for the remaining terms. \square

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